Stabilization of the wave equation with Ventcel boundary conditions

Rémi Buffe

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Abstract

We consider a damped wave equation on a open subset of $\mathbb{R}^n$ or a smooth Riemannian manifold with boundary, with Ventcel boundary conditions, with a linear damping, acting either in the interior or at the boundary. This equation is a model for a vibrating structure with a layer with higher rigidity of thickness $\delta > 0$. By means of a proper Carleman estimate for second-order elliptic operators near the boundary, we derive a resolvent estimate for the wave semigroup generator along the imaginary axis, which in turn yields the logarithmic decay rate of the energy. This stabilization result is obtained uniformly in δ.
KEYWORDS: Stabilization; wave equation; Ventcel boundary conditions; Carleman estimate; resolvent estimate

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1 Introduction, statement of the problem, and main results

1.1 Introduction

We consider a damped wave equation on (Ω, g), a compact Riemannian manifold with smooth boundary $\partial\Omega$, with Ventcel[1] boundary conditions, and we are concerned here with the stabilization of such an equation. This type of boundary conditions is characterized by the presence of a second-order tangential operator at the boundary, for instance the Laplace-Beltrami operator Δ_g^T, reading $\partial_\nu u - \Delta_g^T u = 0$ (ν denotes the outgoing normal unit vector). It generally arises when considering a domain with an thin boundary layer of high rigidity, after some approximations are made. The issue of stabilizing such a wave equation has been the subject of several works, for instance [7, 8, 9, 25, 10, 23, 14]. Here, we consider a linear damping of the form $a(x)\partial_t u$, in the interior of the domain, or $b(x)\partial_t u_{|\partial\Omega}$, within the Ventcel boundary condition at the boundary, where a and b are non-negative functions with a non-empty support of Ω or $\partial\Omega$, respectively. Stabilization is measured by the decay and the convergence to zero of a natural energy function for the solution. Since the seminal works of [27, 1], it is known that a stabilization charaterized by an exponential decay rate for the energy is heavily related to a geometric control condition, GCC for short. Roughly speaking, every generalized geodesic (travelled at speed one), in the sense of [24], needs to meet the control region in a finite time (see also [6]). In our case, we do not impose any condition on the localization of the damping, and we follow the approach of [19, 21, 3] that leads to the proof of a logarithm type decay for the energy. Setting the damped wave equation in a semigroup form, say $\frac{d}{dt}U + AU = 0$, such a decay can be obtained upon deriving of a resolvent estimate for the semigroup generator of the form $\|(i\sigma \operatorname{Id} + A)^{-1}\| \leq C\exp(C|\sigma|)$ for $\sigma \in \mathbb{R}$, with $|\sigma| \geq 1$. Precise statements, including proper operator norms, are given below. Such a resolvent estimate can be achieved from Carleman type estimates for a second-order elliptic operator, taking into account the particular boundary condition used in the definition of the damped wave equation problem. Classical boundary conditions, e.g. homogeneous Dirichlet, homogeneous Neumann in the case of an inner damping (a nonvanishing), or Neumann in the case of a boundary damping (b nonvanishing), were treated in the works cited above. The subject of the present article is to consider Ventcel type condition,

$$\partial_\nu u - \delta\Delta_g^T u + b\partial_t u = 0$$

with a parameter $\delta \in (0, 1]$. In particular, in the result we obtain, the parameter δ is allowed to tend to 0^+ and we recover the result known for Neumann boundary conditions [21]. We refer to Section 1.3 for precise statements.

A large part of the present work is devoted to the proof of a local Carleman estimate near the boundary for the following elliptic problem

$$\begin{cases} \Delta_g u + \sigma^2 u = f & \text{in } \Omega \\ \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} = g & \text{in } \partial\Omega. \end{cases}$$

uniformly in σ and δ for $|\sigma| \geq 1$ and $\delta \in (0, 1]$. Then, this allows us to derive an interpolation inequality leading to the resolvent estimate for the semigroup generator. Using the analysis of [5, 2], we can then obtain the logarithm-decay stabilization result. We also show that a similar result can be obtained dynamical Ventcel type boundary conditions.

[1] The name of Alexander Ventcel is often spelled differently, e.g. Wentzell.

The proof of the Carleman estimate relies on microlocal techniques at the boundary in the spirit of [21, 16, 17, 18, 15]. Near the boudary, our analysis is carried out in normal geodesic coordinates, which eases to use pseudodifferential methods.

The outline of this article is the following. Main results are presented in Section 1.3. In Section 2, we address the well-posedness issues for the damped wave equations we consider as well as the asymptotic behavior of their solutions in the limit $\delta \to 0^+$. Section 3 recalls notions around semiclassical calculus and in Section 4 we describe the local geometry of the problem near the boundary. In preparation for the derivation of the Carleman estimate, various microlocal regions are introduced in Section 5 and microlocal versions of the estimate are obtained in Section 6. These estimates are patched together in Section 7 yielding the desired local Carleman estimate near the boundary. Finally, in Section 8 an interpolation estimate is derived and we achieve the sought resolvent estimate.

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1.2 Statement of the problem

Let (Ω, g) be a Riemannian manifold of dimension n with smooth boundary $\partial\Omega$. For simplicity, Ω is assumed to be connected. The boundary can be seen as a compact Riemannian manifold of dimension $n-1$ without boundary endowed with the induced metric $g_{|\partial\Omega}$. In local coordinates, the gradient and the divergence on (Ω, g) are given by

$$\nabla_g = \sum_{j=1}^{n} g^{ij}\partial_{x_i}, \qquad \operatorname{div}_g u = \frac{1}{\sqrt{\det(g)}} \sum_{j=1}^{n} \partial_{x_j}(\sqrt{\det(g)} u_j).$$

where g^{ij} denotes the coefficients of the inverse of the matrix $g = (g_{ij})_{ij}$, with similar formulae for $\nabla_g^T := \nabla_{g|_{\partial\Omega}}$ and $\operatorname{div}_g^T := \operatorname{div}_{g|_{\partial\Omega}}$. The Laplace-Beltrami operator on (Ω, g) then reads

$$\Delta_g = \operatorname{div}_g \nabla_g = \frac{1}{\sqrt{\det(g)}} \sum_{i,j}^{n} \partial_{x_i}(g^{ij} \sqrt{\det(g)} \partial_{x_j}), \tag{1.1}$$

with a similar formula for Δ_g^T. We shall denote throughout this paper by ν the outgoing unit normal vector to Ω with respect to the Riemannian metric, and ∂_ν the associated normal derivative. In this setting we consider the following wave equation

$$\partial_t^2 u - \Delta_g u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_\nu u_{|\partial\Omega} - \delta \Delta_g^T u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{1.2}$$

which corrresponds to a problem with a static boundary condition of Ventcel type. Dynamic boundary conditions can also be considered, namely

$$\partial_t^2 u - \Delta_g u = 0 \text{ on } \mathbb{R}_t \times \Omega_x \quad \partial_t^2 u_{|\partial\Omega} + \frac{1}{\delta}\partial_\nu u_{|\partial\Omega} - \Delta_g^T u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x. \tag{1.3}$$

In both cases, δ is a small parameter, say $0 < \delta \leq 1$. This kind of boundary conditions may for instance model a thin layer structure surrounding Ω, and the positive parameter δ plays the role of the measure of the thickness of this layer (see Appendix A for a derivation of the model).
We now define usual norms and scalar products on Ω and $\partial\Omega$

$$\left(u, \tilde{u}\right)_{L^2(\Omega)} := \int_\Omega u\bar{\tilde{u}} \mathrm{d}x_g, \qquad \left(v, \tilde{v}\right)_{L^2(\partial\Omega)} := \int_{\partial\Omega} v\bar{\tilde{v}} \mathrm{d}\sigma_g, \tag{1.4}$$

where $\mathrm{d}x_g$ and $\mathrm{d}\sigma_g$ are the volume elements associated with the metrics g and $g_{|\partial\Omega}$. In local coordinates, we have $\mathrm{d}x_g = \sqrt{\det(g)}\mathrm{d}x_1 \ldots \mathrm{d}x_n$, and a similar formula for $\mathrm{d}\sigma_g$. We also introduce the following Sobolev H^1 scalar products

$$\left(u, \tilde{u}\right)_{H^1(\Omega)} = \left(u, \tilde{u}\right)_{L^2(\Omega)} + \left(\nabla_g u, \nabla_g \tilde{u}\right)_{L^2(\Omega)}, \left(v, \tilde{v}\right)_{H^1(\partial\Omega)} = \left(v, \tilde{v}\right)_{L^2(\partial\Omega)} + \left(\nabla_g^T v, \nabla_g^T \tilde{v}\right)_{L^2(\partial\Omega)}. \tag{1.5}$$

Throughout this paper, we shall denote by $||.||$ a norm acting on Ω, and by $|.|$ a norm acting on $\partial\Omega$.

Forming the scalar product of (1.2) with $\partial_t u$ in space and integrating by parts yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u,t) = 0, \text{ with } E(u,t) := \frac{1}{2}\left(||\partial_t u(t)||^2_{L^2(\Omega)} + ||\nabla_g u(t)||^2_{L^2(\Omega)} + \delta|\nabla_g^T u(t)_{|\partial\Omega}|^2_{L^2(\partial\Omega)}\right), \tag{1.6}$$

which corresponds to a conservation of energy $E(u,t)$ of the system.

The purpose of the present article is the study of interior stabilization, namely, the following system

$$\partial_t^2 u - \Delta_g u + a\partial_t u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{1.7}$$

where a is a bounded function of Ω satisfying the condition $a \geq C > 0$ on ω_I, where ω_I is a non empty subset of Ω, as well as the problem with damping affecting a subset the boundary

$$\partial_t^2 u - \Delta_g u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} + b\partial_t u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{1.8}$$

where $b \in W^{1,\infty}(\partial\Omega)$ satisfying $b \geq C > 0$ on a non-empty subset ω_B of $\partial\Omega$. Computing the evolution of the energy as above, we formally obtain respectively

$$E(u,t) - E(u,0) = -\int_0^t \int_\Omega a|\partial_t u|^2, \quad E(u,t) - E(u,0) = -\int_0^t \int_{\partial\Omega} b|\partial_t u_{|\partial\Omega}|^2,$$

which shows that, in both cases, the energy is a non-increasing function of time. We shall prove that the localized damping effect is actually sufficient to ensure that the energy goes to zero at least logarithmically. In [21], in the case where Ω is a ring of $\mathbb{R}^2$, the authors proved that such a logarithmic decay rate is in fact optimal in the case of Neumann boundary conditions.

Below, we shall treat well-posedness and stabilization properties of (1.7) and (1.8) in the same time (Sections 1.3.1 and 2.1). In fact, we shall consider slightly more general operators at the boundary without adding technicality in the analysis. We shall consider the following system

$$\begin{cases} \partial_t^2 u - \Delta_g u + a\partial_t u = 0 & \text{on } \mathbb{R}_t \times \Omega_x \\ \partial_\nu u_{|\partial\Omega} + \delta\Sigma u_{|\partial\Omega} + b\partial_t u_{|\partial\Omega} = 0 & \text{on } \mathbb{R}_t \times \partial\Omega_x, \end{cases} \tag{1.9}$$

where a and b are as above, but at least one is non identically zero, and Σ denotes any positive second-order differential operator on $\partial\Omega$, that vanishes on constant functions, that is

$$\mathbb{C} \subset \ker(\Sigma), \tag{1.10}$$

and which furthermore is self-adjoint for the duality bracket $\langle .,. \rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}$, where the chosen pivot space is $L^2(\partial\Omega)$ endowed with the inner-product defined by (1.4). Note that the definition of $\langle .,. \rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}$ depends on the metric g. Hence there is some connection between the operator Σ and g. In particular $\Sigma = -\Delta_g^T$ is a possible choice for Σ. Observe that H^{-1} is well defined as derivatives of $L^2(\partial\Omega)$ functions in the distribution sense, since $\partial\Omega$ has no boundary. Thus, the bilinear form

$$\left(u_{|\partial\Omega}, u_{|\partial\Omega}\right)_{L^2(\partial\Omega)} + \langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)} \tag{1.11}$$

defines an equivalent norm on $H^1(\partial\Omega)$ to (1.5). Furthermore, we define the energy associated to (1.9)

$$E_s(u,t) := \frac{1}{2}\left(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\nabla_g u(t)\|^2_{L^2(\Omega)} + \delta\left\langle \Sigma u(t)_{|\partial\Omega}, u(t)_{|\partial\Omega}\right\rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}\right). \tag{1.12}$$

The reader should keep in mind that a prototype of such an operator Σ is $-\Delta_g^T$ defined in (1.1), and in this case, the energies E and E_s coincide. To treat the existence and uniqueness properties of evolution system (1.9), it is convenient to recast the problem into a semigroup formalism. Considering the norms appearing in the energy of solutions given in (1.12), we introduce the natural following spaces

$$\mathcal{H}_\delta = \mathcal{V}_\delta \times L^2(\Omega), \quad \delta \in (0,1], \text{ where } \mathcal{V}_\delta = \left\{u \in H^1(\Omega) \,|u_{|\partial\Omega} \in H^1(\partial\Omega)\right\},$$

endowed with the norm

$$\|u\|^2_{\mathcal{V}_\delta} = \|u\|^2_{H^1(\Omega)} + \delta\left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega}\right\rangle_{H^{-1}(\partial\Omega),H^1(\Omega)}. \tag{1.13}$$

The space $\mathcal{V}_\delta$ together with the norm $\|.\|_{\mathcal{V}_\delta}$ has a Hilbert space structure. Observe that this norm is equivalent to $\left(\|.\|^2_{H^1(\Omega)} + \delta|.|^2_{H^1(\partial\Omega)}\right)^{1/2}$. We then define the following norm on $\mathcal{H}_\delta$ as the cannonical norm

$$\|(u,v)\|^2_{\mathcal{H}_\delta} = \|u\|^2_{\mathcal{V}_\delta} + \|v\|^2_{L^2(\Omega)}.$$

Each space $\mathcal{H}_\delta$ and $\mathcal{V}_\delta$ indexed by δ is algebraically equal to $\mathcal{H}_{\delta=1}$ and $\mathcal{V}_{\delta=1}$ respectively. Yet, note that this identification does not hold topologically as δ goes to 0. Next, we define the wave operator

$$A_\delta := \begin{pmatrix} 0 & -\operatorname{Id} \\ -\Delta_g & a(x) \end{pmatrix} \tag{1.14}$$

of domain $D(A_\delta) := \{(u_0, u_1) \mid u_0 \in H^2(\Omega),\ u_{0|\partial\Omega} \in H^2(\partial\Omega),\ u_1 \in \mathcal{V}_\delta,\ \partial_\nu u_{0|\partial\Omega} + \delta\Sigma u_{0|\partial\Omega} + b u_{1|\partial\Omega} = 0\}$. The operator A_δ depends on δ through its domain. In this formalism, system (1.9) reads as an evolution equation

$$\partial_t U + A_\delta U = 0, \tag{1.15}$$

for $U = (u, \partial_t u)$. In the case of dynamic boundary conditions, we shall consider the following problem

$$\partial_t^2 u - \Delta_g u + a\partial_t u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_t^2 u_{|\partial\Omega} + \frac{1}{\delta}\partial_\nu u_{|\partial\Omega} + \Sigma u + \frac{1}{\delta} b\partial_t u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{1.16}$$

where a and b are as in (1.9). Arguing as in (1.6), we define the following energy

$$E_d(u,t) := \frac{1}{2}\left(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\nabla_g u(t)\|^2_{L^2(\Omega)} + \delta|\partial_t u_{|\partial\Omega}(t)|^2_{L^2(\partial\Omega)} + \delta\left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega}\right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)}\right).$$

We shall treat system (1.16) as a system of equations coupled through the normal derivative term with a transmission condition at the boundary

$$\partial_t^2 u - \Delta_g u + a\partial_t u = 0, \quad \partial_\nu u_{|\partial\Omega} + \delta\partial_t^2 y + \delta\Sigma y + b\partial_t y = 0, \quad u_{|\partial\Omega} = y. \tag{1.17}$$

We then define the space of energy

$$\mathcal{K}_\delta := \left\{(u_0, u_1, y_0, y_1) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\partial\Omega) \times L^2(\partial\Omega) \mid u_{0_{|\partial\Omega}} = y_0\right\},$$

endowed with the norm $\|(u_0,u_1,y_0,y_1)\|^2_{\mathcal{K}_\delta} = \|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \delta\left\langle \Sigma y_0, y_0\right\rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)} + \delta|y_1|^2_{L^2(\partial\Omega)}$, yielding a Hilbert space structure. We recast system (1.17) into the evolution equation $\partial_t U + B_\delta U = 0$, where $U = (u, \partial_t u, y, \partial_t y)$, and where B_δ is the operator defined on $\mathcal{K}_\delta$

$$B_\delta := \begin{pmatrix} 0 & -\operatorname{Id} & 0 & 0 \\ -\Delta_g & a & 0 & 0 \\ 0 & 0 & -\operatorname{Id} & 0 \\ \frac{1}{\delta}\gamma_1 & 0 & \Sigma & \frac{1}{\delta}b \end{pmatrix},$$

with domain $D(B_\delta) := \left\{(u_0,u_1,y_0,y_1) \in H^2(\Omega) \times H^1(\Omega) \times H^2(\partial\Omega) \times H^1(\partial\Omega) \mid u_{0_{|\partial\Omega}} = y_0\right\}$. The operator γ_1 denotes here the trace on $\partial\Omega$ of the normal derivative ∂_ν.

1.3 Main results

1.3.1 Stablization results on the damped wave equations

The main results of this article are the following stabilization properties.

Theorem 1.1. *Let $k \geq 1$. There exists $C > 0$ such that for all $0 < \delta \leq 1$ we have the following energy decay estimate*

$$E_s(u,t)^{1/2} \leq \frac{C}{(\log(2+t))^k}\|A_\delta^k U_0\|_{\mathcal{H}_\delta(\Omega)},$$

for all u solutions of (1.9) with initial data $U_0 = (u_0, \partial_t u_0)$.

We also have

Theorem 1.2. *Let $k \geq 1$. There exists $C > 0$ such that for all $0 < \delta \leq 1$ we have the following energy decay estimate*

$$E_d(u,t)^{1/2} \leq \frac{C}{(\log(2+t))^k}\|B_\delta^k U_0\|_{\mathcal{H}_\delta(\Omega)},$$

for all u solutions of (1.16) with initial data $U_0 = (u_0, \partial_t u_0, y_0, \partial_t y_0)$.

Note that zero may be an eigenvalue for both operators A_δ and B_δ associated with vectors of the form $(C,0)$ and $(C,0,C,0)$ respectively, and with assumption (1.10), the energies E_s and E_d are invariant under addition of constants (see Proposition 2.3 and 2.7).

Observe that the decay rate increases as the regularity of the initial data does. In fact, using semi-group properties one can show that if we simply have $E(u,t) \le f(t)E(u,0)$ with $f(t) \to 0$ as $t \to +\infty$ then, in fact, the energy decays exponentially. From [2, 5], it is well known that the stabilization results of Theorem 1.1 and 1.2 can be reduced to deriving the following resolvent estimate along the imaginary axis.

Theorem 1.3. *For all $\sigma \in \mathbb{R}$, $\sigma \neq 0$, the operators $(i\sigma \operatorname{Id} + A_\delta)$ and $(i\sigma \operatorname{Id} + B_\delta)$ are invertible on $\mathcal{H}_\delta$ and $\mathcal{K}_\delta$ respectively. Moreover, there exists C>0 such that*

$$\|(i\sigma \operatorname{Id} + A_\delta)^{-1}\|_{\mathcal{H}_\delta \to \mathcal{H}_\delta} \le Ce^{C|\sigma|} \qquad |\sigma| \ge 1, \tag{1.18}$$

$$\|(i\sigma \operatorname{Id} + B_\delta)^{-1}\|_{\mathcal{K}_\delta \to \mathcal{K}_\delta} \le Ce^{C|\sigma|} \qquad |\sigma| \ge 1. \tag{1.19}$$

As $D(A_\delta)$ is compactly embedded in $\mathcal{H}_\delta$, the spectrum of A_δ is countable. In the case of an undamped wave equation, i.e $a = 0$ and $b = 0$, the operator A_δ is antisymmetric for the inner-product of $\mathcal{H}_\delta$, and then its eigenvalues are purely imaginary. In the case of stabilization (a or b not identically zero), the only eigenvalue on the imaginary axis is zero. Indeed, let $\sigma \in \mathbb{R}^*$, $\sigma \neq 0$ and consider $U = (u_0, u_1)$ satisfying $(A_\delta + i\sigma)U = 0$. This is equivalent to

$$\begin{cases} u_1 - i\sigma u_0 = 0, \qquad -\Delta_g u_0 - ai\sigma u_0 - \sigma^2 u_0 = 0 & \text{in } \Omega, \\ \partial_\nu u_{0|\partial\Omega} + \delta\Sigma u_{0|\partial\Omega} + ib\sigma u_{0|\partial\Omega} = 0 & \text{in } \partial\Omega. \end{cases}$$

Multiplying the second equation by $\overline{u}_0$ and integrating by parts over Ω yields $u_0 = 0$ on ω_I if considering the imaginary part on ω_I, and $u_{0|\partial\Omega} = 0$ on ω_B, thus u_0 satisfies $-\Delta_g u_0 = \sigma^2 u_0$. Thus we can apply Calderón's unique continuation theorem if $\omega_I \neq \emptyset$, and apply Theorem C.1 given in appendix if $\omega_B \neq \emptyset$. The same arguments hold for the operator B_δ. As said above, 0 is an eigenvalue for both operators. To remove this difficulty, we shall work in quotient spaces as described at the end of Sections 2.1.1 and 2.1.2. In these quotient spaces, we can extend the estimates of Theorem 1.3 to $\sigma \in \mathbb{R}$, and (1.18) and (1.19) ensure that all the eigenvalues are not in a closed neighborhood of $i\mathbb{R}$ of the type $\{z := x + iy,\ x \ge 0,\ x \le e^{-C|y|}\}$ (see for instance [21]). This kind of resolvent estimate is heavily related to the Carleman estimate stated in the next section.

1.3.2 Carleman estimate at the boundary

We shall prove Carleman estimates for classes of more general operators in the interior and at the boundary. We thus define the following operators

$$P = -\Delta_g + c(x).\nabla_g + d(x), \qquad S = \Sigma + c^T(x).\nabla_g^T + d^T(x), \tag{1.20}$$

where c (resp. c^T) denotes any L^∞ vector field on Ω (resp. $\partial\Omega$), and d (resp d^T) any L^∞ function on Ω (resp. $\partial\Omega$). The estimate we prove in this paper concerns the following system

$$(P - \sigma^2)u = f \text{ on } \Omega, \quad \partial_\nu u + \delta(S - \kappa\sigma^2)u = g \text{ on } \partial\Omega, \tag{1.21}$$

where σ is a real number, and κ is equal to 0 or 1. The operators $(P - \sigma^2)$ and $(S - \kappa\sigma^2)$ will be denoted by P_σ and S_σ respectively. We introduce the parameter κ in order to prove a Carleman estimate that allows us to treat both cases of static and dynamic boundary conditions at the same time. More precisely, $\kappa = 0$ corresponds to the static case, and $\kappa = 1$ to the dynamic case. Note that in (1.21), we add lower order terms in the interior and at the boundary. Moreover, we consider non-homogeneous equation with f and g as body and surface source terms. To precisely state the result, we need to recallthe notion of sub-ellipticity. For $\tau \ge 1$, we set $P_{\varphi,\sigma} = e^{\tau\varphi}P_\sigma e^{-\tau\varphi}$, where $\varphi \in C^\infty(\mathbb{R}^n)$, and consider $p_{\varphi,\sigma}$ its semi-classical principal symbol. We then have the following definition.

Definition 1.4. *Let V be a bounded open subset of Ω and $\varphi \in C^\infty(\overline{V})$. We say that φ satisfies the sub-ellipticity condition on $\overline{V}$ if there exists $\tau_0 > 0$ such that*

$$p_{\varphi,\sigma}(x,\xi,\tau) = 0 \Longrightarrow \frac{1}{2i}\left\{\overline{p_{\varphi,\sigma}}, p_{\varphi,\sigma}\right\} > 0, \tag{1.22}$$

for all $x \in \overline{V}$, $\xi \in \mathbb{R}^n$, $|\sigma| \ge 1$ and $\tau \ge \tau_0|\sigma|$.

We now consider V a bounded open neighborhood of a point of $\partial\Omega$. We impose additional conditions on φ on V, namely,

$$\nabla_g \varphi \neq 0 \quad \text{on } \overline{V}, \quad \text{and} \quad |\nabla_g^T \varphi| \leq \nu_0 \inf |\partial_\nu \varphi| \quad \text{on } V \cap \partial\Omega, \tag{1.23}$$

for a sufficiently small $\nu_0 > 0$. The local Carleman estimate in the neighborhood of the boundary that we shall prove is stated as following

Theorem 1.5. *Let $x \in \partial\Omega$ and V be an open neighborhood of x in $\overline{\Omega}$. Let φ be a weight function satisfying the conditions (1.22) and (1.23) on $\overline{V}$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\begin{aligned}\tau^3 \|e^{\tau\varphi} u\|^2_{L^2(V)} + \tau \|e^{\tau\varphi} \nabla_g u\|^2_{L^2(V)} + \tau |e^{\tau\varphi} \partial_\nu u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \leq C\Big(\|e^{\tau\varphi} f\|^2_{L^2(V)} \\ + \tau |e^{\tau\varphi} g|^2_{L^2(V\cap\partial\Omega)} + (\delta^2\tau^5 + \tau^3)|e^{\tau\varphi} u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau |e^{\tau\varphi} \nabla_g^T u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)}\Big),\end{aligned} \tag{1.24}$$

and if in addition, $\partial_\nu \varphi(x) < 0$, on $\overline{V}$ we have the stronger estimate

$$\begin{aligned}\tau^3 \|e^{\tau\varphi} u\|^2_{L^2(V)} + \tau \|e^{\tau\varphi} \nabla_g u\|^2_{L^2(V)} + \tau^3 |e^{\tau\varphi} u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau |e^{\tau\varphi} \nabla_g^T u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \\ + \tau |e^{\tau\varphi} \partial_\nu u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \leq C\left(\|e^{\tau\varphi} f\|^2_{L^2(V)} + \tau |e^{\tau\varphi} g|^2_{L^2(V\cap\partial\Omega)}\right),\end{aligned} \tag{1.25}$$

for all $0 < \delta \leq 1$, for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0 |\sigma|$ and for all $u \in C_0^\infty(V)$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ satisfying (1.21).

Observe that the two Carleman estimates are uniform in $\delta > 0$. That will allow us to perform an uniform energy decay estimate with respect to the small parameter δ at the boundary. Furthermore, in the singular limit $\delta \to 0$, we recover the Carleman estimate proved in [20].

Remark 1.6. *Note that this estimate is invariant by adding lower order terms in σ in the following sense: if we set $L := P - \sigma^2 + r(x)\sigma$ and $L^T := S - \sigma^2 + r^T(x)\sigma$, with r and r^T two L^∞ functions, then we can write*

$$\|e^{\tau\varphi} L u\|_{L^2} \leq \|e^{\tau\varphi}(P - \sigma^2)u\|_{L^2} + C\sigma \|e^{\tau\varphi} u\|_{L^2},$$

and the second term can be absorbed by the left hand side of the Carleman estimates of Theorem 1.5 by taking τ_0 large. In the same spirit,

$$|e^{\tau\varphi} L^T u|_{L^2} \leq |e^{\tau\varphi}(S - \sigma^2)u|_{L^2} + C\sigma |e^{\tau\varphi} u|_{L^2},$$

and we can absorb the second term by taking τ_0 sufficiently large. This estimate is also invariant by adding lower order operators. If (1.24) and (1.25) are true for $P = -\Delta_g$, it is also true for P in the form given in (1.20), by taking τ_0 large. It will thus be sufficient to derive these estimates keeping only the prinipal part of P.

2 Well-posedness and asymptotic behavior

In this section, we survey the well-posedness properties of the damped wave equation with static boundary conditions (1.7). We also consider the asymptotic behavior if δ goes to zero. Indeed, formally taking δ equal to zero, system (1.7) becomes a damped wave equation with Neumann boundary conditions. We shall make precise in which spaces such convergence can be proven.

2.1 Well-posedness properties

The well-posedness properties can be stated for general operators. We set

$$\mathcal{A}_\delta := \begin{pmatrix} 0 & -\mathrm{Id} \\ P & a \end{pmatrix},$$

with domain $D(\mathcal{A}_\delta) := \{(u_0, u_1) \mid u_0 \in H^2(\Omega),\ u_{0_{|\partial\Omega}} \in H^2(\partial\Omega),\ u_1 \in \mathcal{V}_\delta(\partial\Omega),\ \partial_\nu u_0 + \delta S u_0 + b u_1 = 0\}$, for P and S be the operators defined by (1.20). In the same idea, we set

$$\mathcal{B}_\delta := \begin{pmatrix} 0 & -\mathrm{Id} & 0 & 0 \\ P & a & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id} \\ \frac{1}{\delta}\gamma_1 & 0 & S & \frac{1}{\delta} b \end{pmatrix},$$

and observe that $D(B_\delta) = D(\mathcal{B}_\delta)$.

2.1.1 The case of static boundary conditions

Proposition 2.1. *There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for all $F \in \mathcal{H}_\delta$, there exists a unique solution $U = (u, v) \in D(\mathcal{A}_\delta)$ of $(\mathcal{A}_\delta + \lambda \,\mathrm{Id})U = F$. Moreover, there exists $C > 0$ such that*

$$\|u\|^2_{H^2(\Omega)} + \delta |u_{|\partial\Omega}|^2_{H^2(\partial\Omega)} + \|v\|^2_{\mathcal{V}_\delta} \leq C\|F\|^2_{\mathcal{H}_\delta},$$

for all $\delta \in (0, 1]$, and $\lambda \geq \lambda_0$.

The proof is given in Apendix B.1. We can now state the existence and uniqueness result for the associated evolution equation.

Proposition 2.2. *Let $U_0 \in D(\mathcal{A}_\delta)$. Then, there exists a unique U in $C^1([0, +\infty), \mathcal{H}_\delta) \cap C([0, +\infty), D(\mathcal{A}_\delta))$ satisfying the Cauchy problem*

$$\partial_t U + \mathcal{A}_\delta U = 0 \text{ for } t \geq 0, \quad U_{t=0} = U_0.$$

where $D(\mathcal{A}_\delta)$ is endowed with the norm of the graph $\|U\|^2_{D(\mathcal{A}_\delta)} = \|U\|^2_{\mathcal{H}_\delta} + \|\mathcal{A}_\delta U\|^2_{\mathcal{H}_\delta}$.

Moreover, we have $\|U(t, .)\|_{\mathcal{H}_\delta} \leq \|U_0\|_{\mathcal{H}_\delta}$ and $\|\partial_t U(t, .)\|_{\mathcal{H}_\delta} \leq \|\mathcal{A}_\delta U_0\|_{\mathcal{H}_\delta}$.

Proof. From the previous propositions, $A_\delta + \lambda_0 \,\mathrm{Id}$ is maximal monotoneous on $\mathcal{H}_\delta$. Then, we can apply the Lumer-Philips theorem (see for instance [26], Theorem 4.3) to obtain the result. □

We now focus on the case $\mathcal{A}_\delta = A_\delta$ (see (1.14)).

Proposition 2.3. *Assume that (1.10) holds. Then*

$$\mathrm{Sp}(A_\delta) \cap i\mathbb{R} = \{0\}, \tag{2.1}$$

and the subspace E_0 formed by the eigenfunctions of A_δ associated with the eigenvalue 0 is

$$E_0 = \mathbb{C}\ {}^t(1, 0). \tag{2.2}$$

Proof. By Proposition 2.1, the spectrum of A_δ is purely discrete. The fact that $i\sigma$, $\sigma \neq 0$ is not an eigenvalue comes from the discussion below Theorem 1.3. It is clear that 0 is an eigenvalue, and that $\mathbb{C}^t(1, 0) \subset E_0$. Let $(u_0, u_1) \in D(A_\delta)$ such that $A_\delta(u_0, u_1) = 0$. We obtain $u_1 = 0$, and thus $-\Delta_g u_0 = 0$. By integration by parts we have

$$\|\nabla_g u_0\|^2_{L^2(\Omega)} + \delta \left\langle \Sigma u_{0|\partial\Omega}, u_{0|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} = 0,$$

and we obtain $\psi = C$ on Ω, which shows equality (2.2). □

Actually, if assumption (1.10) is not satisfied, then 0 is not an eigenvalue for A_δ. Below, we shall work in quotient spaces $\dot{\mathcal{V}}_\delta = \mathcal{V}_\delta/_{E_0}$ and $\dot{\mathcal{H}}_\delta = \mathcal{H}_\delta/_{E_0}$, where $E_0 := \{(C, 0)\ ,\ C \in \mathbb{C}\}$. We set $\dot{A}_\delta$ the operator induced by the projection in the quotient space. We also set: $D(\dot{A}_\delta) := D(A_\delta) \cap \dot{\mathcal{H}}_\delta$. We can endow the space $\dot{\mathcal{V}}_\delta$ with the scalar product

$$\left(u, \tilde{u}\right)_{\dot{\mathcal{V}}_\delta} := \left(\nabla_g u, \nabla_g \tilde{u}\right)_{L^2(\Omega)} + \delta \left\langle \Sigma u, \tilde{u}\right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)},$$

which defines a norm on $\dot{\mathcal{V}}_\delta$, thanks to the Poincaré inequality. For the sake of simplicity, in the sequel we shall do the following abuse of notation: we shall drop the dots and continue to write $\bullet$ in place of $\dot{\bullet}$, where $\bullet$ is one of the spaces above.

Remark 2.4. *Observe moreover that in the case $\mathcal{A}_\delta = A_\delta$, Proposition 2.1 holds with $\lambda_0 = 0$ in the above quotient spaces.*

2.1.2 The case of dynamic boundary conditions

We have the counterpart of proposition 2.1 for the dynamic case.

Proposition 2.5. *There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for all $F \in \mathcal{K}_\delta$, there exists a unique solution $U = (u_0, u_1, y_0, y_1) \in D(B_\delta)$ of $(\mathcal{B}_\delta + \lambda \,\mathrm{Id})U = F$. Moreover, there exists $C > 0$ such that*

$$\|u_0\|^2_{H^2(\Omega)} + \delta |y_0|^2_{H^2(\partial\Omega)} + \|u_1\|^2_{H^1(\Omega)} + \delta |y_1|^2_{H^1(\partial\Omega)} \leq C\|F\|^2_{\mathcal{K}_\delta},$$

for all $\delta \in (0, 1]$, and $\lambda \geq \lambda_0$.

The proof is given in Appendix B.2. This leads to the following well-posedness result for the damped wave equation (1.17) written in a semigroup setting.

Proposition 2.6. *Let $U_0 \in D(\mathcal{B}_\delta)$. Then there exists a unique U in $C^1([0,+\infty),\mathcal{K}_\delta) \cap C([0,+\infty), D(\mathcal{B}_\delta))$ satisfying the Cauchy problem*

$$\partial_t U + \mathcal{B}_\delta U = 0, \quad U_{t=0} = U_0$$

where $D(\mathcal{B}_\delta)$ is endowed with the norm of the graph $\|U\|^2_{D(\mathcal{B}_\delta)} = \|U\|^2_{\mathcal{K}_\delta} + \|\mathcal{B}_\delta U\|_{\mathcal{K}_\delta}$.

Moreover, we have $\|U(t,.)\|^2_{\mathcal{K}_\delta} \le \|U_0\|^2_{\mathcal{K}_\delta}$ and $\|\partial_t U(t,.)\|^2_{\mathcal{K}_\delta} \le \|\mathcal{B}_\delta U_0\|^2_{\mathcal{K}_\delta}$.

We state the following result about the eigenvalues of the operator B_δ.

Proposition 2.7. *Assume that (1.10) holds. Then*

$$\mathrm{Sp}(B_\delta) \cap i\mathbb{R} = \{0\}, \tag{2.3}$$

and the subspace F_0 formed by the eigenfunctions of B_δ associated with the eigenvalue 0 is

$$F_0 = \mathbb{C}\ {}^t(1,0,1,0).$$

Proof. It is sufficient to repeat the proof of Proposition 2.3. □

We then quotient the space $\mathcal{K}_\delta$ by F_0, and still denote it $\mathcal{K}_\delta$ by abuse of notation.

2.2 Asymptotic behavior as δ goes to zero

In this section, we study the asymptotic behavior as δ goes to zero of the solutions u_δ of the damped wave system with static Ventcel boundary condition

$$\begin{aligned} &\partial_t^2 u_\delta - \Delta_g u_\delta + a\partial_t u_\delta = f_\delta \text{ in } \Omega\times\mathbb{R}, \qquad \partial_\nu u_{\delta|\partial\Omega} - \delta\Delta_g^T u_{\delta|\partial\Omega} + b\partial_t u_{\delta|\partial\Omega} = 0 \text{ in } \partial\Omega\times\mathbb{R}, \\ &(u_\delta(0), \partial_t u_\delta(0)) = U_\delta^0 \text{ in } \Omega, \end{aligned} \tag{2.4}$$

to the solution of the damped wave equation with homogeneous Neumann boundary condition

$$\partial_t^2 v - \Delta_g v + a\partial_t v = f \text{ in } \Omega, \qquad \partial_\nu v_{|\partial\Omega} + b\partial_t v_{|\partial\Omega} = 0 \text{ in } \partial\Omega, \qquad (v(0),\partial_t v(0)) = V^0 \text{ in } \Omega. \tag{2.5}$$

Proposition 2.8. *Assume $f_\delta \rightharpoonup f$ in $L^2(0,T;L^2(\Omega))$ and $U_\delta^0 = V^0 \in \mathcal{H}$. We then obtain*

$$u_\delta \rightharpoonup v \text{ in } L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

and we have the estimate at the boundary $|\nabla_g^T u_{\delta|\partial\Omega}|_{L^2(\partial\Omega)} = O(\delta^{-1/2})$.

The proof is given in Appendix C.1.

3 Notation and semi-classical operators

In the sections below, we shall use the following notation: $\mathbb{R}^n \ni x = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$, and $\mathbb{R}^n \ni \xi = (\xi', \xi_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ and we shall consider the operators $D = -i\partial_x$ and $D' = -i\partial_{x'}$. For V a neighborhood of a point of the boundary $\partial\Omega$ (resp. a neighborhood of 0 in $\mathbb{R}^n$), we set $V^+ = V\cap\Omega$ (resp. $V^+ = V\cap\mathbb{R}^n$), where $\mathbb{R}^n_+$ is the half-space $\{x\in\mathbb{R}^n,\ x_n > 0\}$.

3.1 Semi-classical operators acting on $\mathbb{R}^n$

Here we recall some facts on semi-classical pseudo-differential operators with a large parameter τ, say $\tau \ge \tau_0 \ge 1$. We shall denote by $\mathcal{S}^m_\tau$ the space of smooth functions $a(x,\xi,\tau)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$, with $\tau\ge\tau_0$ as a large parameter, that satisfy the following behavior at infinity: for all multi-indices α,β there exists $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi,\tau)\right| \le C_{\alpha,\beta}(\tau^2 + |\xi|^2)^{(m-|\beta|)/2},$$

for all $(x, \xi', \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times [\tau_0, +\infty)$. For $a \in \mathcal{S}^m_\tau$, we define pseudo-differential operator of order m, denoted by $A = \mathrm{Op}(a)$:

$$Au(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.\xi} a(x, \xi, \tau)\hat{u}(\xi)\mathrm{d}\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

One says that a is the symbol of A. We shall denote Ψ^m_τ the set of pseudo-differential operators of order m and denote by $\sigma(A)$ (resp. $\sigma(a)$) the principal symbol of the operator A (resp. the symbol a) and thus $\sigma(D) = \xi$. We shall also denote by $\mathcal{D}^m_\tau$ the space of semi-classical differential operators, i.e the case when the symbol $a(x, \xi, \tau)$ is a polynomial function of order m in (ξ, τ). Throughout the article, we shall use the following order function on the whole phase-space: $\lambda_\tau = (\tau^2 + |\xi|^2)^{1/2}$. We recall here the composition formula of pseudo-differential operators. Let $a \in \mathcal{S}^m_\tau$ and $b \in \mathcal{S}^{m'}_\tau$, $m, m' \in \mathbb{R}$, we have

$$\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(a\#b),$$

for some $a\#b \in \mathcal{S}^{m+m'}_\tau$, and for all $N \in \mathbb{N}$, there exists $R_N \in \mathcal{S}^{m+m'-N}_\tau$ such that

$$a\#b(x, \xi, \tau) = \sum_{|\alpha|\leq N} \frac{1}{i^{|\alpha|}\alpha!} \partial^\alpha_\xi a(x, \xi, \tau)\partial^\alpha_x b(x, \xi, \alpha) + R_N(x, \xi, \tau). \tag{3.1}$$

For a review on symbolic calculus we refer the reader to [12].

3.2 Tangential semi-classical operators

In the section we consider pseudo-differential operators which only acts in the tangential direction x', with parameter x_n. We define $\mathcal{S}^m_{T,\tau}$ as the set of smooth functions $a(x, \xi', \tau)$ defined for τ as a large parameter, say $\tau \geq \tau_0 \geq 1$, satisfying the following behavior at infinity: for all multi-indices $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^{n-1}$ there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_{\xi'} a(x, \xi', \tau)\right| \leq C_{\alpha,\beta}(\tau^2 + |\xi'|^2)^{(m-|\beta|)/2},$$

for all $(x, \xi', \tau) \in \mathbb{R}^n_+ \times \mathbb{R}^{n-1} \times [\tau_0, +\infty)$. For $a \in \mathcal{S}^m_{T,\tau}$, we define a tangential pseudo-differential operator $B := \mathrm{Op}_T(b)$ of order m by

$$Bu(x) := \frac{1}{(2\pi)^{n-1}} \iint_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} e^{i(x'-y').\xi'} b(x, \xi', \tau)u(y', x_n)\mathrm{d}y'\mathrm{d}\xi'$$

As in the previous section, we define $\Psi^m_{T,\tau}$ as the set of tangential pseudo-differential operators of order m, and $\mathcal{D}^m_{T,\tau}$ the set of tangential differential operators of order m. We shall denote the tangential order function by $\lambda_{T,\tau} := (\tau^2 + |\xi'|^2)^{1/2}$, and define the following semi-classical Sobolev tangential norms, for fonctions on $\mathbb{R}^{n-1}$ or traces of functions on $\mathbb{R}^n$ at $\{x_n = 0\}$

$$|u|_{m,\tau} := |\,\mathrm{Op}_T(\lambda^m_{T,\tau})u|_{L^2(\mathbb{R}^{n-1})}.$$

We also define the following semi-classical norms on the half space $\mathbb{R}^n_+$

$$\|u\|^2_{m,\tau} := \sum_{k=0}^{m} \|D^k_n \,\mathrm{Op}_T(\lambda^{m-k})u\|^2_{L^2(\mathbb{R}^n_+)}.$$

Observe that , if $m \in \mathbb{N}$, this semi-classical norm is equivalent to $\sum_{|\alpha|\leq m} \tau^{|\alpha|}\|D^{m-|\alpha|}u\|_{L^2(\mathbb{R}^n_+)}$, uniformly for $\tau \in [\tau_0, +\infty)$. Below, we shall use several times the following trace lemma.

Lemma 3.1. *There exists $C > 0$ such that for all $u \in S(\mathbb{R}^n)$, for $\tau \geq 1$,*

$$|u_{|_{x_n=0}}|_0 \leq C\tau^{-1/2}\|u\|_{1,\tau}. \tag{3.2}$$

The proof is left to the reader.

Remark 3.2. *In this paper, we shall use operators whose symbol depends on a additional parameter σ, say $a(x, \xi, \tau, \sigma)$, such that they satisfy*

$$\left|\partial^\alpha_x \partial^\beta_\xi a(x, \xi, \tau, \sigma)\right| \leq C_{\alpha,\beta}(\sigma^2 + \tau^2 + |\xi|^2)^{(m-|\beta|)/2}.$$

However, in the region where $\tau \gtrsim |\sigma|$, we have $a \in \mathcal{S}^m_\tau$, and this property will be used several times.

4 Local setting in the neighborhood of the interface

Here, we consider normal geodesic coordinates $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ in a neighborhood V of a point of the boundary. Locally, we have $\Omega = \{x_n > 0\}$ and $\partial\Omega = \{x_n = 0\}$. We recall that from the remark below Theorem 1.5, we can consider only the higher order terms in the operator P, since the form of the estimates we want to prove is insensitive to the addition of low order terms. In such local coordinates, the principal part of operator P takes the form (and we still denote by P, by abuse of notation)

$$P = D_n^2 + R(x, D'), \qquad R(x, D') = \sum_{j,k=1}^{n-1} D_j(a_{j,k}(x) D_k),$$

where $a_{j,k} = a_{k,j}$ and, if we denote by $r(x, \xi')$ the homogeneous principal symbol of R,

$$r(x, \xi') = \sum_{j,k=1}^{n-1} a_{j,k}(x)\xi_j\xi_k \in \mathbb{R} \text{ and } \exists C > 0,\ \forall (x, \xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1},\ r(x, \xi') \geq C|\xi'|^2. \tag{4.1}$$

Note that the principal part of the operator P is chosen to be formally self-adjoint. We also denote the homogeneous principal symbol of P by $p(x, \xi) = \xi_n^2 + r(x, \xi')$. Whenever V is a neighborhood of 0 in $\mathbb{R}^n$, we shall denote by $\overline{C_0^\infty}(V^+)$ the space of restrictions to $\overline{V}^+ := \mathbb{R}^n \cap \{x_n > 0\}$ of C^∞ functions on $\mathbb{R}^n$ compactly supported in V. On the boundary $\{x_n = 0\}$, the operator S is an elliptic second-order differential operator in the x'-direction. If we denote by $s(x', \xi')$ its homogeneous principal symbol, we have

$$s(x', \xi') \in \mathbb{R} \text{ and } \exists C > 0,\ \forall (x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, s(x', \xi') \geq C|\xi'|^2.$$

Observe that in these local coordinates, we have $\partial_\nu = -\partial_{x_n}$, where ∂_ν denote the outgoing normal derivative associated with the metric g. In what follows, we shall denote $P_\sigma := P - \sigma^2$ and $S_\sigma := S - \kappa\sigma^2$ for all $\sigma \in \mathbb{R}$, and $\kappa \in \{0, 1\}$.

4.1 Operator conjugaison by a weight function

As is done classically, we introduce the following conjugated operator

$$P_{\varphi,\sigma} = e^{\tau\varphi} P_\sigma e^{-\tau\varphi},$$

of homogeneous principal symbol $p_{\varphi,\sigma}$, for a smooth function φ to be precisely defined below. We denote also $S_{\varphi,\sigma}$ the conjugated boundary operator

$$S_{\varphi,\sigma} = e^{\tau\varphi_{|x_n=0}} S_\sigma e^{-\tau\varphi_{|x_n=0}},$$

of homogeneous principal symbol $s_{\varphi,\sigma}$. In what follows, we set: $v = e^{\tau\varphi}u$, and we thus have $e^{\tau\varphi} P_\sigma u = P_{\varphi,\sigma} v$ and $e^{\tau\varphi_{|x_n=0}} S_\sigma u_{|x_n=0} = S_{\varphi,\sigma} v_{|x_n=0}$. We have $P = P_2 + iP_1$ by setting

$$P_2 = \frac{1}{2}\left(P_{\varphi,\sigma} + P_{\varphi,\sigma}^*\right), \qquad P_1 = \frac{1}{2i}\left(P_{\varphi,\sigma} - P_{\varphi,\sigma}^*\right).$$

Note that P_2 and P_1 are formally self-adjoints. Observing that $e^{\tau\varphi} D_j e^{-\tau\varphi} = D_j + i\tau\partial_{x_j}\varphi$, we have

$$P_2 = D_n^2 - (\tau\partial_{x_n}\varphi)^2 - \sigma^2 + R(x, D') - r(x, \tau \mathrm{d}_{x'}\varphi) = P_\sigma - p(x, \tau \mathrm{d}_x\varphi), \tag{4.2}$$

$$P_1 = \tau D_n \partial_{x_n}\varphi + \tau\partial_{x_n}\varphi D_n + \tau \sum_{j,k=1}^{n} (D_j a_{j,k}(x)\partial_k\varphi + a_{j,k}(x)\partial_j\varphi D_k) \tag{4.3}$$

$$= 2\tau(\partial_{x_n}\varphi D_n + \tilde{r}(x, \mathrm{d}_{x'}\varphi, D')) \mod \tau\mathscr{D}_T^0, \tag{4.4}$$

where $\tilde{r}$ denotes the symmetric bilinear form associated with the quadratic form r, $\tilde{r}(x, \xi, \eta) = \sum_{j,k=1}^{n} a_{j,k}(x)\xi.\eta$. At the boundary, we also have $S_{\varphi,\sigma} = S_2 + iS_1$, with

$$S_2 = \frac{1}{2}\left(S_{\varphi,\sigma} + S_{\varphi,\sigma}^*\right) \qquad S_1 = \frac{1}{2i}\left(S_{\varphi,\sigma} - S_{\varphi,\sigma}^*\right),$$

that are formally self adjoints and of the form

$$S_2 = s(x, D') - s(x', \tau \mathrm{d}_{x'}\varphi_{|x_n=0}) - \kappa\sigma^2 \mod \mathcal{D}^1_{T,\tau} \text{ and } S_1 = 2\tau\tilde{s}(x', \mathrm{d}_{x'}\varphi_{|x_n=0}, D') \mod \mathcal{D}^1_{T,\tau}, \tag{4.5}$$

where $\tilde{s}$ denotes the symmetric bilinear form associated with the quadratic form s. Their principal symbols are respectively

$$p_2(x,\xi,\tau,\sigma)) = -\sigma^2 + \xi_n^2 + r(x,\xi') - \tau^2(\partial_{x_n}\varphi)^2 - r(x, \tau \mathrm{d}_{x'}\varphi), \tag{4.6}$$

$$p_1(x,\xi,\tau) = 2\tau\left(\partial_{x_n}\varphi\xi_n + \tilde{r}(x, \mathrm{d}_{x'}\varphi, \xi')\right), \tag{4.7}$$

$$s_2(x',\xi',\tau,\sigma) = s(x,\xi') - s(x', \tau\mathrm{d}_{x'}\varphi_{|x_n=0}) - \kappa\sigma^2, \tag{4.8}$$

$$s_1(x',\xi',\tau) = 2\tau\tilde{s}(x', \mathrm{d}_{x'}\varphi_{|x_n=0}, \xi'). \tag{4.9}$$

Note that $p_{\varphi,\sigma}(x,\xi,\tau) = p_2(x,\xi,\tau,\sigma) + ip_1(x,\xi,\tau)$. We shall denote the tangential parts of the symbols p_2 and p_1 by

$$\tilde{p}_2(x,\xi',\tau,\sigma) = -\sigma^2 + r(x,\xi') - p(x,\tau\mathrm{d}_x\varphi), \tag{4.10}$$

$$\tilde{p}_1(x,\xi',\tau) = 2\tau\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi). \tag{4.11}$$

With this notation, if u satisfies $\partial_\nu u_{|x_n=0} + \delta S_\sigma(x', D')u_{|x_n=0} = \tilde{\Theta}$, then v satisfies

$$D_n v_{|x_n=0} = K v_{|x_n=0} + \Theta \tag{4.12}$$

where $K \in \delta\mathcal{D}^2_{T,\tau} + \tau\mathcal{D}^0_{T,\tau}$, with principal symbol

$$\begin{aligned} k(x',\xi',\tau,\sigma) &= 2\tau\delta\tilde{s}(x',\xi',\mathrm{d}_{x'}\varphi) - i\left(\tau\partial_{x_n}\varphi + \delta s(x',\xi') - \delta s(x',\tau\mathrm{d}_{x'}\varphi) - \delta\kappa\sigma^2\right) \\ &= \delta s_1(x',\xi',\tau) - i(\tau\partial_{x_n}\varphi + s_2(x',\xi',\tau\sigma)), \end{aligned} \tag{4.13}$$

and $\Theta = ie^{\tau\varphi}\tilde{\Theta}$, recalling that $\partial_\nu = -\partial_{x_n}$.

Under the action of conjugaison by the weight function, the resulting operator $P_{\varphi,\sigma}$ is not elliptic. In order to handle the presence of the characteristics set, we shall impose the following condition on the weight function which ensures the positivity of some commutators.

Definition 4.1. *Let V be a bounded open set of $\mathbb{R}^n$. We say the weight function $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies the sub-ellipticity property in $\overline{V}$ if $|d_x\varphi| > 0$ in $\overline{V}$ and if there exist $C > 0$ and $\tau_0 > 0$ such that for $|\sigma| \geq 1 > 0$*

$$\forall (x,\xi) \in \overline{V}\times\mathbb{R}^n,\ \forall\tau \geq \tau_0|\sigma|,\ \ p_{\varphi,\sigma}(x,\xi,\tau) = 0 \Longrightarrow \{p_2,p_1\}(x,\xi,\tau) \geq C\lambda_\tau^3. \tag{4.14}$$

Here, we state the sub-ellipticity condition in normal geodesic coordinates. However, note that this condition is geometrically invariant, and thus this definition is equivalent to (1.22) (observe that by the homogeneity of the Poisson bracket, $\{p_2, p_1\} > 0$ implies (4.14), see the end of the proof of Proposition 4.2). The following proposition provides a construction of a weight function φ that yields sub-ellipticity using a classical convexification procedure. A proof without the parameter σ can be found in [16]. With the parameter σ, a proof is given in Appendix D.1.

Proposition 4.2. *Let V be a bounded open subset of $\mathbb{R}^n$ and $\psi \in C^\infty(\mathbb{R}^n)$ such that $|d_x\psi| \geq C > 0$ on $\overline{V}$.Then, there exists $\lambda > 0$ sufficiently large, such that the function $\varphi := e^{\lambda\psi}$ satisfies the sub-ellipticity condition on $\overline{V}$ for $\tau \geq \tilde{C}|\sigma|$, where $\tilde{C}$ is a constant satisfying $\tilde{C} \geq \dfrac{1}{\lambda \inf\varphi}$.*

Observe that we impose τ to be larger than σ here. This condition appears naturally in the following proof. In what follows, τ will thus be the principal parameter. Inspecting the proof we actually obtain the stronger property: $p_2 = 0 \Rightarrow \{p_2, p_1\} \geq C\lambda_\tau^3$.

Considering the previous proposition, we shall often write $\tau \geq \tau_0|\sigma|$, where $\tau_0 > 0$ is taken sufficiently large, and we shall use the fact that $\tau + \sigma \approx \tau$ on many occasions in what follows.

4.2 Weight function properties

In this section, we first recall the required properties (1.22), (1.23) for the function φ to be an admissible weight function on $\overline{V}$, where V is an bounded open neighborhood of 0 in $\mathbb{R}^n$. Yet, here we states these conditions in the normal geodesic coordinates introduced above, and we provide a construction for such a function. The weight function to be used, $\varphi \in C^\infty(V)$, is chosen so as to satisfy the following conditions

- $|\nabla_x \varphi| \geq C > 0$;
- For a given $\nu_0 > 0$, we have
$$|\partial_{x_j}\varphi| \leq \nu_0 \inf_{\overline{V}} |\partial_{x_n}\varphi| \quad j = 1, \ldots, n-1 \tag{4.15}$$
- φ satisfies the sub-ellipticity condition (4.14), on $\overline{V}$, which is given in Section 4.1.

The value of $\nu_0 > 0$ will be determined in Lemma 5.3 and in Lemma 6.7 and it is meant to be small. With this parameter, we enforce the weight function to be relatively flat in the tangential directions as compared to its variations in the normal direction. In the applications we have in mind, we shall use weights of the form $e^{\lambda\psi}$. The two first conditions are satisfied if

- $|\nabla\psi| \geq C > 0$;
- For a given $\nu_0 > 0$, we have $|\partial_{x_j}\psi| \leq \nu_0 \inf_{\overline{V}} |\partial_{x_n}\psi| \quad j = 1 \ldots n-1$.

If $|\nabla\psi| \geq C > 0$ then for λ sufficiently large, the third condition is satisfied (see Proposition 4.2). Observe that if φ is an admissible weight function fulfilling the above conditions, then its normal derivative cannot be zero, implying: $|\partial_{x_n}\varphi| \geq C > 0$ on $\overline{V}$.

4.3 A boundary quadratic form

Using integrations by parts and symbolic calculus, we derive a first estimate. It exhibits a quadratic form involving the two traces $u_{|_{x_n=0}}$ and $\partial_{x_n} u_{|_{x_n=0}}$ at the boundary. This estimate is central in what follows. Actually, we shall exploit its structure when considering the phase-space region where the operator $P_{\sigma,\varphi}$ is not elliptic, and use the sub-ellipticity condition (4.14) in a crucial way. This estimate is now classical and is proved in [20]. A proof with the parameter σ is given in Appendix D.2.

Proposition 4.3. *Let V be an open neighborhood of 0 in $\mathbb{R}^n$ and let φ be a weight function satisfying the sub-ellipticity condition (4.14) in $\overline{V}^+$, and assume that $|\partial_{x_n}\varphi| \geq C > 0$ on $\overline{V}$. Then there exists $\tau_0 > 0$ and $C' > 0$ such that*

$$C'\tau\|v\|_{1,\tau}^2 + \tau \operatorname{Re} \mathcal{B}(v) \leq \|P_{\varphi,\sigma}v\|_{0,\tau}^2$$

for all $v \in \overline{C}_0^\infty(V^+)$, $|\sigma| \geq 1$ and $\tau \geq \tau_0|\sigma|$, where

$$\mathcal{B}(v) = 2\Big(\partial_{x_n}\varphi D_n v_{|_{x_n=0}}, D_n v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})} + \Big(A_1 v_{|_{x_n=0}}, D_n v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})}$$
$$+ \Big(D_n v_{|_{x_n=0}}, A_1' v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})} + \Big(A_2 v_{|_{x_n=0}}, v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})}.$$

The operators A_1, A_1' and A_2 are differential, and

- *A_1, $A_1' \in \mathcal{D}^1_{T,\tau}$ and satisfy*
$$a_1 := \sigma(A_1) = \sigma(A_1') = 2\tilde{r}(x, \xi', d_{x'}\varphi); \tag{4.16}$$
- *$A_2 \in \mathcal{D}^2_{T,\tau}$ and satisfies*
$$a_2 := \sigma(A_2) = 2\partial_{x_n}\varphi\Big(\sigma^2 + p(x, \tau d_{x'}\varphi) - r(x, \xi')\Big). \tag{4.17}$$

5 Microlocal regions and roots properties

Here, we consider the principal symbol of the conjugated operator (see (4.6)-(4.11))

$$\begin{aligned} p_{\varphi,\sigma}(x, \xi, \tau) &= p_2(x, \xi, \tau, \sigma) + ip_1(x, \xi, \tau) \\ &= \tilde{p}_2(x, \xi', \tau, \sigma) + i\tilde{p}_1(x, \xi', \tau) + \xi_n^2 + 2i\tau\partial_{x_n}\varphi\xi_n \\ &= (\xi_n + i\tau\partial_{x_n}\varphi)^2 + (\tau\partial_{x_n}\varphi)^2 + \tilde{p}_2(x, \xi', \tau, \sigma) + i\tilde{p}_1(x, \xi', \tau). \end{aligned}$$

We set: $m = (\tau\partial_{x_n}\varphi)^2 + \tilde{p}_2(x,\xi',\tau,\sigma) + i\tilde{p}_1(x,\xi',\tau)$. Then, we can write $p_{\varphi,\sigma}$ as a factorized second-order polynomial function in the ξ_n variable

$$p_{\varphi,\sigma}(x,\xi',\xi_n,\tau) = (\xi_n + i\alpha + i\tau\partial_{x_n}\varphi)(\xi_n - i\alpha + i\tau\partial_{x_n}\varphi),$$

where $\alpha \in \mathbb{C}$ satisfies $\mathrm{Re}(\alpha) \geq 0$ and $\alpha^2 = m$. We can write

$$p_{\varphi,\sigma}(x,\xi',\xi_n,\tau) = (\xi_n - \rho^+)(\xi_n - \rho^-),$$

with $\rho^- = -i\tau\partial_{x_n}\varphi - i\alpha$ and $\rho^+ = -i\tau\partial_{x_n}\varphi + i\alpha$. Observe that there exists $C > 0$ such that

$$|\rho^\pm| \leq C\lambda_T, \tau. \tag{5.1}$$

We set $\mu(x,\xi',\tau,\sigma) = \tilde{p}_2(x,\xi',\tau,\sigma) + \dfrac{\tilde{p}_1(x,\xi',\tau)^2}{(2\tau\partial_{x_n}\varphi)^2}$. Note that it is a homogeneous function of degree 2 in the (ξ',τ,σ) variable.

Lemma 5.1. *We have the following:*

if $\mu(x,\xi',\tau,\sigma) < 0$ *, then $\rho^\pm \notin \mathbb{R}$ and* $\mathrm{sign}(\mathrm{Im}(\rho^-)) = \mathrm{sign}(\mathrm{Im}(\rho^+)) = \mathrm{sign}(-\partial_{x_n}\varphi)$*;*

if $\mu(x,\xi',\tau,\sigma) = 0$ *, then*

- *if $\partial_{x_n}\varphi > 0$, then $\rho^+ \in \mathbb{R}$ and* $\mathrm{Im}(\rho^-) < 0$*; thus* $(x,\xi',\rho^+,\tau) \in \mathrm{Char}(P_{\varphi,\sigma})$
- *if $\partial_{x_n}\varphi < 0$, then $\rho^- \in \mathbb{R}$ and* $\mathrm{Im}(\rho^+) > 0$*; thus* $(x,\xi',\rho^-,\tau) \in \mathrm{Char}(P_{\varphi,\sigma})$*;*

if $\mu(x,\xi',\tau,\sigma) > 0$ *, then $\rho^\pm \notin \mathbb{R}$,* $\mathrm{Im}(\rho^-) < 0$ *and* $\mathrm{Im}(\rho^+) > 0$*;*

The different root configurations are represented in figure 1. From Lemma 5.2, we have $\mu < 0$ implies $|\xi'| \lesssim \tau$, and that $\mu > 0$ implies $|\xi'| \gtrsim \tau$, and that $\mu = 0$ implies $\tau \lesssim |\xi'| \lesssim \tau$.

Proof. For $z = a + ib \in \mathbb{C}$, $a \neq 0$, we have

$$\mathrm{Re}(z^2) = a^2 - \frac{\mathrm{Im}(z^2)^2}{4a^2}. \tag{5.2}$$

Using (5.2) with $z = \alpha$, observe that

$$\begin{aligned}\mu &= \mathrm{Re}(m) - (\tau\partial_{x_n}\varphi)^2 + \frac{\mathrm{Im}(m)^2}{(2\tau\partial_{x_n}\varphi)^2} = \mathrm{Re}(\alpha)^2 - (\tau\partial_{x_n}\varphi)^2 + \frac{\mathrm{Im}(\alpha^2)^2}{4}\left(\frac{1}{(\tau\partial_{x_n}\varphi)^2} - \frac{1}{\mathrm{Re}(\alpha)^2}\right)\\ &= \left(\mathrm{Re}(\alpha)^2 - (\tau\partial_{x_n}\varphi)^2\right)\left(1 + \frac{\mathrm{Im}(\alpha^2)^2}{4\,\mathrm{Re}(\alpha)^2(\tau\partial_{x_n}\varphi)^2}\right).\end{aligned} \tag{5.3}$$

Thus $\mu \gtreqqless 0$ if and only if $\mathrm{Re}\,\alpha - \tau|\partial_{x_n}\varphi| = |\mathrm{Re}\,\alpha| - \tau|\partial_{x_n}\varphi| \gtreqqless 0$. This allows us to conclude as $\mathrm{Im}\,\rho^\pm = \pm\mathrm{Re}\,\alpha - \tau\partial_{x_n}\varphi$. $\square$

The sign of μ is related to the value of the tangential variables $|\xi'|$ with respect to τ.

Lemma 5.2. *For $\delta_0 > 0$ taken sufficiently small, there exists $C > 0$ such that we have the following:*

if $\mu(x,\xi',\tau,\sigma) \leq \delta_0\lambda_{T,\tau}^2$ *, then $|\xi'|^2 \leq C\tau^2$;*

if $\mu(x,\xi',\tau,\sigma) \geq -\delta_0\lambda_{T,\tau}^2$ *, then $\frac{\tau^2}{C} \leq |\xi'|^2$.*

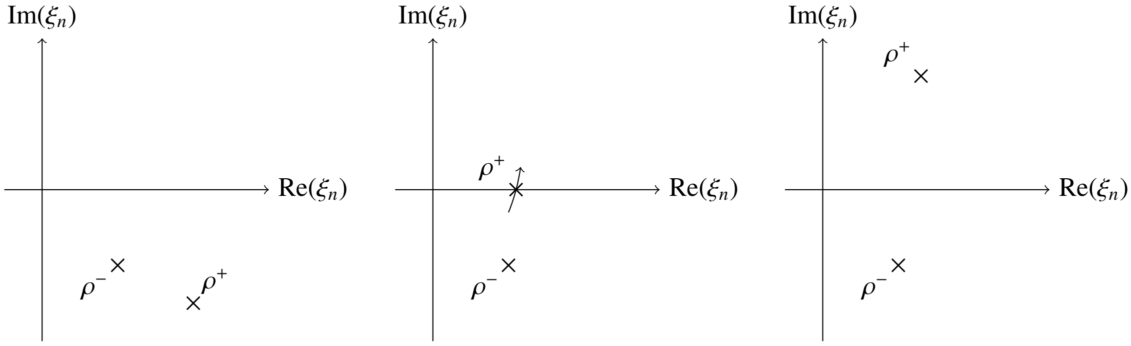
Proof. Suppose first that $\mu(x,\xi',\tau,\sigma) \leq \delta_0\lambda_{T,\tau}^2$. This means

$$r(x,\xi') + \frac{\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi)^2}{(\partial_{x_n}\varphi)^2} \leq \delta_0\lambda_{T,\tau}^2 + p(x,\tau\mathrm{d}_x\varphi) + \sigma^2,$$

implying for some $C_0 > 0$, and $C_1 > 0$, we have $C_0|\xi'|^2 \leq \delta_0|\xi'|^2 + C_1\tau^2$. Thus, for $\delta_0 < C_0$, we have $|\xi'| \lesssim \tau^2$. Suppose second that $\mu(x,\xi',\tau,\sigma) \geq -\delta_0\lambda_{T,\tau}^2$, meaning

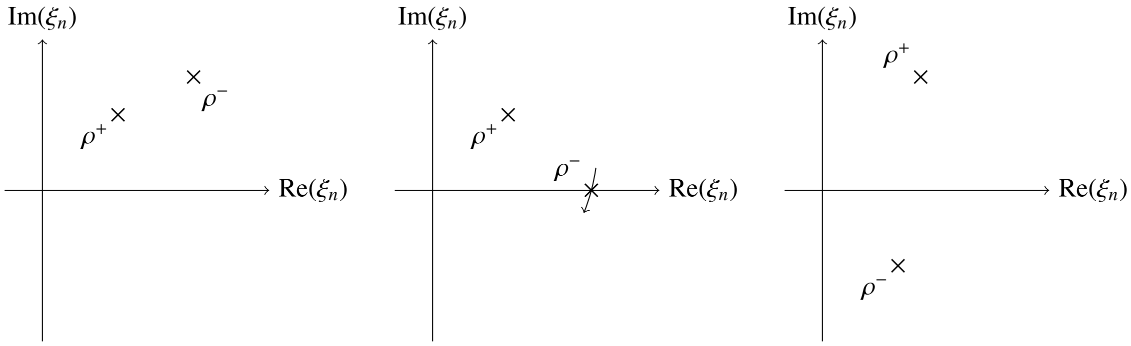
$$p(x,\tau\mathrm{d}_x\varphi) + \sigma^2 \geq r(x,\xi') + \delta_0\lambda_{T,\tau}^2 + \frac{\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi)^2}{(\partial_{x_n}\varphi)^2}.$$

The case where $\partial_{x_n}\varphi > 0$:



$\mathcal{E}^-$: $\mu < 0$ $\quad$ $\mathcal{E}^0$: $\mu = 0$, ρ^+ crosses the real axis $\quad$ $\mathcal{E}^+$: $\mu > 0$

The case where $\partial_{x_n}\varphi < 0$:



$\mathcal{E}^-$: $\mu < 0$ $\quad$ $\mu = 0$, ρ^- crosses the real axis $\quad$ $\mathcal{E}^+$: $\mu > 0$

Figure 1: Position of the roots of $p_{\varphi,\sigma}$ as μ varies.

This implies that for some $C_2 > 0$, and $C_3 > 0$ we have $C_2\tau^2 + \sigma^2 \leq C_3|\xi'|^2 + \delta_0\tau^2$. Thus for $\delta_0 < C_2$, we have $\tau^2 \lesssim |\xi'|^2$. □

In the case $\mu = 0$, the operator $P_{\varphi,\sigma}$ is not elliptic, as one of the roots ρ^+ or ρ^- is real. There is a (real) characteristic set. The ellipticity or the non-ellipticity of $P_{\varphi,\sigma}$ can thus be expressed through an algebraic condition on the tangential variables. We introduce the following phase-space regions

$$\mathcal{E}^+ = \left\{(x, \xi', \tau, \sigma) \in V^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R} \mid |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma| \mid \mu(x, \xi', \tau, \sigma) > \eta_1\lambda_{T,\tau}^2\right\}$$

$$\mathcal{E}^- = \left\{(x, \xi', \tau, \sigma) \in V^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R} \mid |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma| \mid \mu(x, \xi', \tau, \sigma) < -\eta_1\lambda_{T,\tau}^2\right\}$$

$$\mathcal{E}^0 = \left\{(x, \xi', \tau, \sigma) \in V^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R} \mid |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma| \mid -2\eta_1\lambda_{T,\tau}^2 < \mu(x, \xi', \tau, \sigma) < 2\eta_1\lambda_{T,\tau}^2\right\},$$

where $\eta_1 > 0$ will be chosen sufficiently small below (see Proposition 6.5 and Lemma 6.7). These microlocal regions are sketched in Figure 2. We shall thus cut the tangential phase-space into three pieces to isolate the different behaviors of the roots of $P_{\varphi,\sigma}$.

Lemma 5.3. *(Localization of the characteristic sets). Let φ be a weight function satisfying the properties of Section 4.2. Then, there exist $C > 0$, $C_0 > 0$ and $\tau_0 > 0$ such that for all $|\sigma| \geq 1$ and $\tau \geq \tau_0|\sigma|$ we have*

$$\mathrm{Re}\, s_{\varphi,\sigma}(x', \xi', \tau) = s(x', \xi') - s(x', \tau d_{x'}\varphi_{|x_n=0}) - \kappa\sigma^2 \geq C\lambda_{T,\tau}^2 \qquad \text{if} \quad \mu(x, \xi', \tau, \sigma) \geq -C_0\lambda_{T,\tau}^2.$$

In particular, we have the following inclusions:

$$\mathrm{Char}(S_{\varphi,\sigma}) \subset \mathrm{Char}(\mathrm{Re}(S_{\varphi,\sigma})) \subset \{\mu \leq -C_0\lambda_{T,\tau}^2\} \cap \{x_n = 0\} \subset \mathcal{E}^- \cap \{x_n = 0\},$$

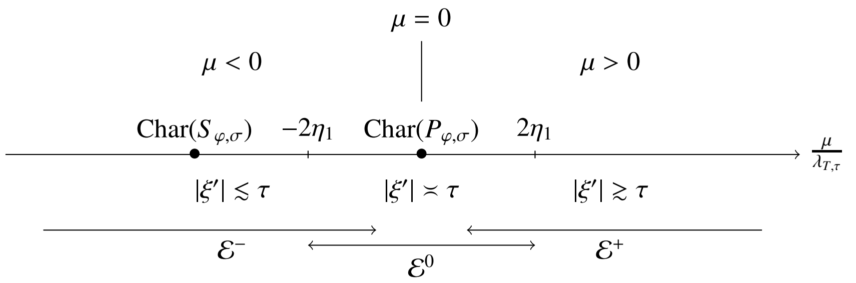


Figure 2: Representation of the three microlocal regions.

if $0 < \eta_1 \leq C_0$.

Proof. We have, on the one hand

$$\begin{aligned} s(x', \xi') - s(x', \tau \mathrm{d}_{x'} \varphi_{|x_n=0}) - \kappa\sigma^2 &\geq C''|\xi|^2 - C'''\tau^2 |\mathrm{d}_{x'}\varphi_{|x_n=0}|^2 - \kappa\sigma^2 \\ &\geq C''|\xi'|^2 - C''' \nu_0^2 \tau^2 \inf_{\overline{V}} |\partial_{x_n}\varphi|^2 - \kappa\sigma^2, \end{aligned}$$

and on the other hand, from Lemma 5.2, $|\xi'|^2 \gtrsim \lambda_{T,\tau}$, if $\mu \geq -C_0 \lambda_{T,\tau}^2$ for $C_0 > 0$ sufficiently small. This yields

$$\begin{aligned} s(x', \xi') - s(x', \tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2 &\geq C'''' \left(|\xi'|^2 + \sigma^2 + \tau^2 \right) - C'' \nu_0^2 \tau^2 \inf |\partial_{x_n}\varphi|^2 - \kappa\sigma^2 \\ &\gtrsim \lambda_{T,\tau}^2, \end{aligned}$$

for all $\tau \geq \tau_0|\sigma|$, by taking ν_0 sufficiently small and τ_0 sufficiently large. The other statements follow. □

Here, we used that the weight function is chosen sufficiently flat in the tangential directions with respect to the normal one in a crucial way: it ensure that the two characteristic sets, that of $P_{\varphi,\sigma}$ and that of $S_{\varphi,\sigma}$ are associated with two different microlocal regions. We shall derive three microlocal estimates corresponding to the previous regions determined by the sign of μ, and prove an uniform Carleman estimate with respect to the small parameter δ, that appears in the boundary condition in (1.21). In fact, if we only want estimates with a fixed δ, say equal to one, we can prove such an estimate using only two microlocal regions $\mathcal{E}^-$ and $\mathcal{E}^0 \cup \mathcal{E}^+$. This is due to the fact that the principal symbol of the boundary operator is of order 2 and elliptic in high frequencies. Then, if $\delta = 1$, only second orders terms are relevant. Here, because δ varies in $(0, 1]$, we have to treat second- and first-order terms. That is precisely the reason of the apparition of our particular treatment in the zone $\mathcal{E}^+$.

Lemma 5.4. *Let* $\chi \in \mathcal{S}^0_{T,\tau}$ *homogeneous of degree 0, such that* $\mathrm{supp}(\chi) \subset \mathcal{E}^+$. *Then,* $\chi\rho^\pm \in \mathcal{S}^1_{T,\tau}$ *and there exists* $C > 0$ *such that* $|\operatorname{Im}(\rho^\pm)| \geq C\lambda_{T,\tau}$ *on the support of* χ.

Proof. Let us show first $\chi\rho^\pm \in \mathcal{S}^1_{T,\tau}$. As $\tau\partial_{x_n}\varphi \in \mathcal{S}^1_{T,\tau}$, it suffices to prove that $\chi\alpha \in \mathcal{S}^1_{T,\tau}$. We have $\chi\frac{|\alpha^2|}{\lambda^2_{T,\tau}} \in \mathcal{S}^0_{T,\tau}$ as a homogeneous function of degree 0 in the (ξ', τ, σ) variable. In the region $\mathcal{E}^+$, we claim that there exists a neighborhood U of $\overline{\mathbb{R}^-}$ such that $m \notin U$. Indeed, consider $(x, \xi', \tau, \sigma) \in \mathcal{E}^+ \cap (V \times \mathbb{S}_{(\xi',\tau,\sigma)=1})$ where $\mathbb{S}_{(\xi',\tau,\sigma)}$ denotes the unit sphere in the variable (ξ', τ, σ). Suppose that $\operatorname{Im}(m) = 0$. This implies that $\tilde{p}_1(x, \xi', \tau) = 0$ and thus $\mu(x, \xi', \tau, \sigma) = \tilde{p}_2(x, \xi', \tau, \sigma)$. The definition of $\mathcal{E}^+$ yields that $\tilde{p}_2(x, \xi', \tau, \sigma) > 0$ and thus $\operatorname{Re}(m) > 0$. By a compacity argument we have there exists a constant $C > 0$ such that $\operatorname{Re}(m) \geq C > 0$, and then the claim is proved by homogeneity. This allows us to define $\frac{\alpha}{\lambda_{T,\tau}} = F(\frac{m}{\lambda^2_{T,\tau}})$, where F is the complex principal square root. Using Theorem 18.1.10 in [12], we obtain $\frac{\alpha}{\lambda_{T,\tau}} \in \mathcal{S}^0_{T,\tau}$ in a conic neighborhood of the support of χ. Now, we show that $|\operatorname{Im}(\rho^\pm)| \geq C\lambda_{T,\tau} > 0$. We have

$$|\operatorname{Im}(\rho^\pm)| \geq |\operatorname{Re}(\alpha)| - \tau|\partial_{x_n}\varphi|. \tag{5.4}$$

Using (5.3) and observing that on the support of χ, we have $\mu \gtrsim C\lambda^2_{T,\tau}$, we obtain $\operatorname{Re}(\alpha)^2 - (\tau\partial_{x_n}\varphi)^2 \gtrsim \lambda^2_{T,\tau}$, and with (5.4) this concludes the proof.

□

6 Microlocal Carleman estimates

We recall that the operators $P_{\varphi,\sigma}$ and K are defined in Section 4.1. In each region, we define cut-off functions χ depending on (x, ξ', τ, σ), but from Remark 3.2, the parameter σ will not be involved in the symbolic calculus.

6.1 Estimate in $\mathcal{E}^+$

In this region, we have $\mu > \eta_1$ and thus $|\xi'| \gtrsim \tau$ and both operators $P_{\varphi,\sigma}$ and $S_{\varphi,\sigma}$ are elliptic, and this allows us to estimate v in the interior and at the boundary from a single observation at the boundary. Observe that the estimate here is of better quality than that in the other zones $\mathcal{E}^-$ and $\mathcal{E}^0$.

Proposition 6.1. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and φ satisfying the conditions of section 4.2, and $\chi(x,\xi',\tau,\sigma) \in \mathcal{S}^0_{T,\tau}$ be such that $\operatorname{supp}(\chi) \subset \mathcal{E}^+$. Then there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\tau^2 \| \operatorname{Op}_T(\chi) v\|^2_{1,\tau} + (\tau + \delta^2\tau^3) | \operatorname{Op}_T(\chi) v_{|x_n=0}|^2_{1,\tau} + \tau |D_n \operatorname{Op}_T(\chi) v_{|x_n=0}|^2_{0,\tau} \leq C \Big(\|P_{\varphi,\sigma} v\|^2_{0,\tau} \\ + \tau |(D_n - K) v_{|x_n=0}|^2_{0,\tau} + \|v\|^2_{1,\tau} + \tau^{-1} |D_n v_{|x_n=0}|^2_{0,\tau} + \tau^{-1} |v_{|x_n=0}|^2_{1,\tau} \Big) \quad (6.1)$$

for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0 |\sigma|$, for all $v \in \overline{C}^\infty_0(V^+)$, and $\delta \in (0,1]$.

Proof. In this microlocal region, we shall apply the Calderón projector method. We shall denote

$$P_{\varphi,\sigma} v = \tilde{f} \text{ in } \mathbb{R}^n_+, \qquad D_n v_{|x_n=0} - K v_{|x_n=0} = \Theta \text{ in } \{x_n = 0\},$$

for $v \in \overline{C}^\infty_0(V^+)$. Let $\chi \in \mathcal{S}^0_{T,\tau}$ as in the statement of the proposition. We set $w := \operatorname{Op}_T(\chi) v$ and $g := \operatorname{Op}_T(\chi)\tilde{f}$. Hence

$$w_1 := P_{\varphi,\sigma} w = \operatorname{Op}_T(\chi) P_{\varphi,\sigma} v + [P_{\varphi,\sigma}, \operatorname{Op}_T(\chi)] v = g + [P_{\varphi,\sigma}, \operatorname{Op}_T(\chi)] v,$$

and on $\{x_n = 0\}$ we have

$$(D_n - K) w_{|x_n=0} =: w_0 = [D_n - K, \operatorname{Op}_T(\chi)] v_{|x_n=0} + \operatorname{Op}_T(\chi) \Theta_{|x_n=0}, \quad (6.2)$$

and as $[D_n - K, \operatorname{Op}_T(\chi)] \in \delta \Psi^1_{T,\tau} + \Psi^0_{T,\tau}$ we find

$$|w_0|_0 \lesssim \delta |v_{|x_n=0}|_{1,\tau} + |v_{|x_n=0}|_0 + |\Theta|_0. \quad (6.3)$$

Observing that the commutator $[P_{\varphi,\sigma}, \operatorname{Op}_T(\chi)] \in \Psi^1_\tau$ and does not depend on σ, we obtain

$$\|w_1\|_{L^2} \lesssim \|g\|_{L^2} + \|v\|_{1,\tau}. \quad (6.4)$$

In what follows, we shall denote by $\underline{w}$ the extension of w by 0 on $\{x_n < 0\}$. We thus obtain the following equality on the whole $\mathbb{R}^n$:

$$P_{\varphi,\sigma} \underline{w} = \underline{w}_1 - i\gamma_1(w) \delta_{x_n=0} - \gamma_0(w) \delta'_{x_n=0} + 2\tau \partial_{x_n} \varphi \gamma_0(w) \delta_{x_n=0} \quad (6.5)$$

where $\gamma_0(w) = w_{|x_n=0}$, $\gamma_1(w) = (D_n w)_{|x_n=0}$, and δ is the Dirac measure. Recalling that ρ^+ and ρ^- are the two complex roots of the principal symbol $p_{\varphi,\sigma}$ viewed as a polynomial in the variable ξ_n (with $\operatorname{Im} \rho^+ > 0$ and $\operatorname{Im} \rho^- < 0$), we find $-2i\tau \partial_{x_n} \varphi = \rho^+(x,\xi',\tau,\sigma) + \rho^-(x,\xi',\tau,\sigma)$. With this relation, (6.5) reads

$$P_{\varphi,\sigma} \underline{w} = \underline{w}_1 + W_0 \delta_{x_n=0} + W_1 \delta'_{x_n=0}, \quad (6.6)$$

where

$$W_1 = -\gamma_0(w), \qquad W_0 = i \left(\operatorname{Op}(\rho^+ + \rho^-) \gamma_0(w) - \gamma_1(w) \right), \quad (6.7)$$

Let U_0 and U_1 be two conic neighborhoods of $\operatorname{supp}(\chi)$ in $(V \cap \mathbb{R}^n_+) \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}$ such that $\overline{U_1} \subset U_0$ and $\overline{U_0} \subset \mathcal{E}^+$. We also define, for $\tau_0 > 0$:

$$V_0 := \left\{ (x,\xi,\tau,\sigma) \mid x \in V^+,\ |\sigma| \geq 1,\ \tau \geq \tau_0 |\sigma|,\ |\xi_n| \geq C_0 |(\xi',\tau)| \right\},$$

$$V_1 := \left\{ (x,\xi,\tau,\sigma) \mid x \in V^+, |\sigma| \geq 1, \tau \geq \tau_0 |\sigma|,\ |\xi_n| \geq C_1 |(\xi',\tau)| \right\},$$

for $0 < C_1 < C_0$ chosen sufficiently large. Note that V_0 and V_1 are conic in (ξ,τ,σ). Let $\hat{\chi}(x,\xi,\tau,\sigma) \in \mathcal{S}^0_\tau$ (see Remark 3.2), homogeneous of degree 0, be such that $\hat{\chi}$ is equal to 1 on the conic set $(U_1 \times \mathbb{R}) \cup V_0$, and is equal to 0 outside $(U_0 \times \mathbb{R}) \cup V_1$. Note that it is possible since $((U_1 \times \mathbb{R}) \cup V_0) \cap \mathbb{S}_{|(\xi,\tau,\sigma)|=1} \Subset ((U_0 \times \mathbb{R}) \cup V_1) \cap \mathbb{S}_{|(\xi,\tau,\sigma)|=1}$, where $\mathbb{S}_{|(\xi,\tau,\sigma)|=1}$ denotes the unit sphere on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}$. The microlocal neighborhoods are represented in Figure 3.

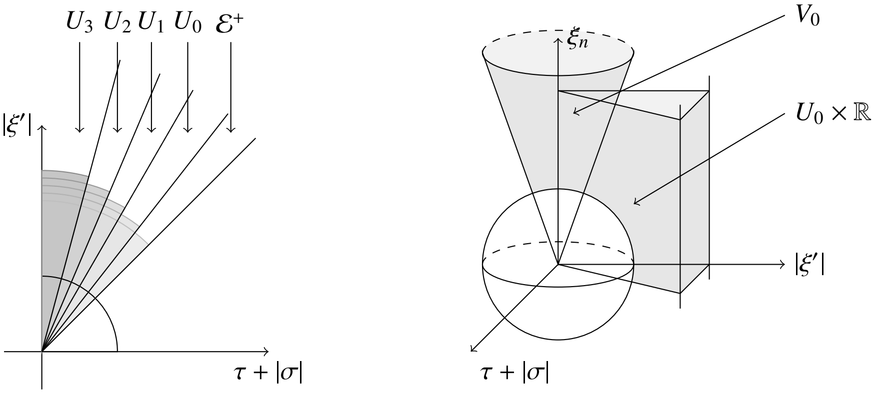


Figure 3: Representation of the different conic neighborhoods.

Observe moreover that on $\operatorname{supp}(\hat{\chi})$, we have $p_{\varphi,\sigma} \neq 0$. Indeed, on the one hand it is true on $U_1 \times \mathbb{R}$ since it is true on $\mathcal{E}^+ \times \mathbb{R}$, and on the other hand it is true on V_1 since $p_{\varphi,\sigma} = 0$ is equivalent to $\xi_n = \rho^+$ or $\xi_n = \rho^-$, and implies from (5.1), that there exists a constant $C > 0$ such that $|\xi_n| \le C\lambda_{T,\tau}$, which can be avoided i n V_1, for C_1 chosen sufficiently large. Thus, we can construct a parametrix $E_N = \operatorname{Op}(e) \in \Psi_\tau^{-2}, N \in \mathbb{N}$, such that

$$e = \sum_{j=0}^{N} e_j, \quad e_0 = \frac{\hat{\chi}}{p_{\varphi,\varphi}}, \quad e_j \in \Psi_\tau^{-2-j}$$

with e_j homogeneous of dergree $-2 - j$ and satisfying $E_N P_{\varphi,\sigma} = \operatorname{Op}(\hat{\chi}) + R_N$. where $R_N \in \Psi_\tau^{-N}$. From (6.6) we find

$$\underline{w} = E_M \left(W_1 \delta'_{x_n=0} + W_0 \delta_{x_n=0} \right) + g_1, \tag{6.8}$$

where $g_1 = E_N(\underline{w}_1) + (\operatorname{Id} - \operatorname{Op}(\hat{\chi}))\, \underline{w} - R_N \underline{w}$. As in $\{x_n > 0\}$ we have $w = \operatorname{Op}(\chi)v$, we observe that $\operatorname{supp}(1 - \hat{\chi}) \cap (\operatorname{supp}(\chi) \times \mathbb{R}) = \emptyset$, and we shall make use of the following lemma of [12], Theorem 18.1.35, which proof can be adapted to the semi-classical setting we consider here.

Lemma 6.2. *Let $a_{T,m}(x, \xi', \tau, \sigma) \in \mathcal{S}^m_{T,\tau}$ and $b_{m'}(x, \xi, \tau, \sigma) \in \mathcal{S}^{m'}_\tau$ and assume that for some $\delta > 0$ we have $b_{m'}(x, \xi, \tau, \sigma) = 0$ if $\delta|\xi_n| > 1$ and $|(\xi', \tau)| \le \delta|\xi_n|$. If moreover, $(\operatorname{supp}(a_{T,m} \times \mathbb{R})) \cap \operatorname{supp}(b_{m'}) = \emptyset$, then*

$$\operatorname{Op}_T(a_{T,m}) \circ \operatorname{Op}(b_{m'}) \in \bigcap_{M \in \mathbb{N}} \Psi_\tau^{-M}, \quad \textit{and} \quad \operatorname{Op}(b_{m'}) \circ \operatorname{Op}_T(a_{T,m}) \in \bigcap_{M \in \mathbb{N}} \Psi_\tau^{-M}.$$

With Lemma 6.2 we find $(\operatorname{Id} - \operatorname{Op}(\hat{\chi})) \circ \operatorname{Op}(\chi) \in \bigcap_M \Psi_\tau^{-M}$, which yields:

$$\|g_1\|_{2,\tau} \lesssim \|v\|_{1,\tau} + \|g\|_{0,\tau}. \tag{6.9}$$

We have

$$\begin{cases} E_N \left(W_1 \delta'_{x_n=0} + W_0 \delta_{x_n} \right) = T_0 W_0 + T_1 W_1 \\ T_j W := \operatorname{Op}_T(t_j)(x_n) W = \dfrac{1}{(2\pi)^{n-1}} \displaystyle\int_{\mathbb{R}^{n-1}} e^{i(x'-y').\xi'} t_j(x, \xi', \tau, \sigma) W(y') dy' d\xi' \\ t_j = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} e^{i x_n \xi_n} (i\xi_n)^j e(x, \xi, \tau, \sigma) d\xi_n. \end{cases} \tag{6.10}$$

Note that from (5.1), $e(x, \xi, \tau, \sigma) \lesssim \lambda_\tau^{-2}$ for all $\xi_n \in \mathbb{C}$ satisfying $|\xi_n| \ge R\lambda_{T,\tau}$, with R sufficiently large. Then t_0 defines an absolutely convergent integral, but t_1 has to be taken in the oscillatory integral sense, and

$$t_1 = \frac{1}{2\pi} \partial_{z_n} \int_{\mathbb{R}} e^{i z_n \xi_n} e(x, \xi, \tau, \sigma) d\xi_n \Big|_{z_n = x_n}, \tag{6.11}$$

where the derivative is taken in the distribution sense. As for $|\xi_n| \ge C_1 \lambda_{T,\tau}$, $\hat{\chi} = 1$, the symbol $e(x, \xi, \tau, \sigma)$ is holomorphic in the variable ξ_n, we can thus change the contour of integration:

$$t_0 = \frac{1}{2\pi} \int_\beta e^{i x_n \xi_n} e(x, \xi, \tau, \sigma) d\xi_n, \tag{6.12}$$

where $\beta = \{\xi_n \in \mathbb{R} \mid \xi_n \in [-R\lambda_{T,\tau}, R\lambda_{T,\tau}]\} \cup \{\xi_n \in \mathbb{C},\ |\xi_n| = R\lambda_{T,\tau},\ \operatorname{Im}\xi_n \geq 0\}$, with $R > 0$ chosen sufficiently large to have $\rho^{\pm}$ in the complex domain delimited by β (such a constant R exists since $\rho^{\pm} \in \mathcal{S}^1_{T,\tau}$). From (6.11), we can also define:

$$t_1 = \frac{1}{2\pi}\int_{\beta} e^{ix_n\xi_n} i\xi_n e(x,\xi,\tau,\sigma)d\xi_n. \tag{6.13}$$

For a review of oscillatory integrals, the reader may refer to [11, Section 7.8]. Observe that $\operatorname{Op}_T(t_j)(x_n)$ is a x_n-family of pseudo-differential operators acting on $\mathbb{R}^{n-1}$. In fact, from (6.12) and (6.13) we have

$$\left|\partial^l_{x_n}\partial^{\alpha_1}_{x'}\partial^{\alpha_2}_{\xi'} t_j\right| \leq C_{l,\alpha_1,\alpha_2}\lambda_{T,\tau}^{-1-|\alpha_2|+l+j} \qquad j = 0, 1. \tag{6.14}$$

Now let U_2 and U_3 be two conic neighborhoods of $\operatorname{supp}(\chi)$ in $(V \cap \mathbb{R}^n_+) \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}$ such that $\overline{U}_3 \subset U_2$ and $\overline{U}_2 \subset U_0$, and we choose $\chi_1(x,\xi',\tau,\sigma) \in \mathcal{S}^0_{T,\tau}$ such that $\chi_1 = 1$ on U_2 and $\chi_1 = 0$ outside U_3 (see figure 3). In addition, we set $s_j = \chi_1 t_j$ and $g_2 = \operatorname{Op}_T((1-\chi_1)t_0)(x_n)W_0 + \operatorname{Op}_T((1-\chi_1)t_1)(x_n)W_1$ which yields

$$\underline{w} = \operatorname{Op}(s_0)(x_n)W_0 + \operatorname{Op}(s_1)(x_n)W_1 + g_3, \tag{6.15}$$

where $g_3 = g_1 + g_2$. By tangential symbolic calculus (the normal variable ξ_n is not involved in the calculus), as $\operatorname{supp}(1-\chi_1) \cap \operatorname{supp}(\chi) = \emptyset$, using (6.14), the trace formula (3.2), we obtain

$$\|\operatorname{Op}_T(\lambda^l_{T,\tau})D^{l'}_{x_n}g_2\|_{0,\tau} \lesssim \|v\|_{1,\tau} + \tau^{-1}|\gamma_1(v)|_{0,\tau}, \tag{6.16}$$

for all $l \in \mathbb{R}$, $l' \in \mathbb{N}$. This allows us to estimate g_3:

$$\|g_3\|_{2,\tau} \lesssim \|g\|_{0,\tau} + \|v\|_{1,\tau} + \tau^{-1}|\gamma_1(v)|_{0,1}. \tag{6.17}$$

We now estimate s_j. The symbol $e(x,\xi',\xi_n,\tau,\sigma)$ is holomorphic in ξ_n on the support of χ_1, we then can change the contour of integration in the complex plane

$$s_j = \frac{\chi_1(x,\xi',\tau,\sigma)}{2\pi}\int_{\beta_0} e^{ix_n.\xi_n} e(x,\xi',\xi_n,\tau,\sigma)(i\xi_n)^j d\xi_n \qquad j = 0, 1 \tag{6.18}$$

where β_0 is a direct contour surrounding ρ^+ in the region where $\operatorname{Im}\xi_n \geq c_0\lambda^1_{T,\tau}$, for some $c_0 > 0$. (note that it is possible from Section 5). By the residue formula, we have

$$e^{-ix_n\rho^+}s_j = i^{j+1}\frac{(\rho^+)^j\chi_1}{\rho^+ - \rho^-} + m_j, \tag{6.19}$$

with $m_j \in \mathcal{S}^{-2+j}_{T,\tau}$. With (6.18), we can estimate:

$$|D^l_{x_n}\partial^{\alpha_1}_{x'}\partial^{\alpha_2}_{\xi'}s_j| \leq C_{l,\alpha_1,\alpha_2}e^{-c_0x_n\lambda_{T,\tau}}(|\xi'| + \tau)^{-1+j+l-|\alpha_2|} \qquad j = 0, 1.$$

We thus obtain $e^{c_0x_n\tau}D^l_{x_n}s_j$ is bounded in $\mathcal{S}^{-1+j+l}_{T,\tau}$ uniformly in $x_n \geq 0$. This yields

$$\begin{aligned}
\|\operatorname{Op}_T(\lambda_{T,\tau})\operatorname{Op}_T(s_j)(x_n)W_j\|^2_{L^2(\mathbb{R}^n_+)} &\leq C\int_{x_n>0}|\operatorname{Op}_T(s_j)W_j|^2_{1,\tau}(x_n)dx_n \\
&\leq C\int_{x_n>0} e^{-2c_0x_n\tau}|e^{c_0x_n\tau}\operatorname{Op}_T(s_j)W_j|^2_{1,\tau}(x_n)dx_n \\
&\leq C|W_j|^2_{j,\tau}\int_{x_n>0} e^{-2c_0x_n\tau}dx_n \\
&\leq C\tau^{-1}|W_j|^2_{j,\tau},
\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}
\|D_n\operatorname{Op}_T(s_j)(x_n)W_j\|^2_{L^2(\mathbb{R}^n_+)} &\leq C\int_{x_n>0}|D_n\operatorname{Op}_T(s_j)W_j|^2_0(x_n)dx_n \\
&\leq C\int_{x_n>0} e^{-2c_0x_n\tau}|e^{c_0x_n\tau}D_n\operatorname{Op}_T(s_j)W_j|^2_0(x_n)dx_n \\
&\leq C|W_j|^2_{j,\tau}\int_{x_n>0} e^{-2c_0x_n\tau}dx_n \\
&\leq C\tau^{-1}|W_j|^2_{j,\tau}.
\end{aligned} \tag{6.21}$$

Using (6.15), (6.17), (6.20) and (6.21), we obtain

$$\begin{aligned}\|w\|_{1,\tau} &\lesssim \tau^{-1}\|g_3\|_{2,\tau} + \tau^{-1/2}|W_1|_{1,\tau} + \tau^{-1/2}|W_0|_0 + \tau^{-2}|D_n v|_{L^2} \\ &\lesssim \tau^{-1}\|g\|_{0,\tau} + \tau^{-1}\|v\|_{1,\tau} + \tau^{-1/2}|W_1|_{1,\tau} + \tau^{-1/2}|W_0|_{0,\tau} + \tau^{-2}|\gamma_1(v)|_{L^2}. \end{aligned} \tag{6.22}$$

It remains to determine the two traces W_0 et W_1. Taking the trace $x_n = 0^+$ in (6.15) and using the definition of W_0 and W_1 in (6.7):

$$\gamma_0(w) = \mathrm{Op}_T(a)\gamma_0(w) + \mathrm{Op}_T(b)\gamma_1(w) + \gamma_0(g_3), \tag{6.23}$$

with $a \in \mathcal{S}^0_{T,\tau}$, of principal symbol $\sigma(a) = -\chi_1 \frac{\rho^-}{\rho^+-\rho^-}|_{x_n=0}$ and $b \in \mathcal{S}^{-1}_{T,\tau}$, of principal symbol $\sigma(b) = \frac{\chi_1}{\rho^+-\rho^-}|_{x_n=0}$ (see (6.19)). Here, $\mathrm{Op}_T(a)$ and $\mathrm{Op}_T(b)$ are the so-called Calderón projectors. Moreover, using again the trace formula (3.2), the remainder $\gamma_0(g_3)$ satisfies

$$|\gamma_0(g_3)|_{1,\tau} \le \tau^{-1/2}\|g_3\|_{2,\tau} \lesssim \tau^{-1/2}\|g\|_{0,\tau} + \tau^{-1/2}\|v\|_{1,\tau} + \tau^{-3/2}|D_n v_{|x_n=0}|_{0,\tau}. \tag{6.24}$$

The principal symbol of b satisfies $|\sigma(b)| \ge C\lambda_{T,\tau}^{-1}$ in $\mathrm{supp}(\chi_1)$. Let $\tilde{\chi}(x', \xi', \tau, \sigma) \in \mathcal{S}^0_{T,\tau}$ satisfy the same hypothesis than χ_1, and such that $\chi_1 = 1$ on the support of $\tilde{\chi}$. We can thus construct a parametrix, of symbol denoted by $l_1 \in \mathcal{S}^1_{T,\tau}$, such that

$$\mathrm{Op}_T(l_1)\,\mathrm{Op}_T(b) = \mathrm{Op}_T(\tilde{\chi}) + \tilde{R}, \qquad \tilde{R} \in \mathcal{S}^{-\infty}_\tau.$$

Moreover the principal symbol of l_1 is given by $\tilde{\chi}(\rho^+ - \rho^-)$. We now apply this parametrix to (6.23), and we find

$$\mathrm{Op}_T(l_1)\,\mathrm{Op}_T(1-a)\gamma_0(w) = \gamma_1(w) + g_4, \tag{6.25}$$

where $g_4 = S_1\gamma_1(v) + S_0\gamma_0(v) + \mathrm{Op}_T(l_1)\gamma_0(g_3)$, with $S_0, S_1 \in \mathcal{S}^{-\infty}_{T,\tau}$. Here we used

$$\begin{aligned}\mathrm{Op}_T(\tilde{\chi})\gamma_1(w) &= \gamma_1(w) - \mathrm{Op}_T(1-\tilde{\chi})\gamma_1(w) \\ &= \gamma_1(w) - \mathrm{Op}_T(1-\tilde{\chi})D_n\,\mathrm{Op}(\chi)v_{|x_n=0} \\ &= \gamma_1(w) - \mathrm{Op}_T(1-\tilde{\chi})\,\mathrm{Op}(\chi)_{|x_n=0}D_n v_{|x_n=0} - \mathrm{Op}_T(1-\tilde{\chi})\left([D_n, \mathrm{Op}_T(\chi)]v\right)_{|x_n=0},\end{aligned}$$

and $\mathrm{Op}_T(1-\tilde{\chi}) \circ \mathrm{Op}(\chi)_{|x_n=0} \in \mathcal{S}^{-\infty}_{T,\tau}$, and $\mathrm{Op}_T(1-\tilde{\chi})[D_n, \mathrm{Op}_T(\chi)]_{|x_n=0} \in \mathcal{S}^{-\infty}_{T,\tau}$. From (6.24), we have the following estimate on g_4

$$\begin{aligned}|g_4|_{0,\tau} &\le |S_1\gamma_1(v)|_{0,\tau} + |S_0\gamma_0(v)|_{0,\tau} + |\gamma_0(g_3)|_{1,\tau}, \\ &\lesssim \tau^{-1/2}\|g\|_{0,\tau} + \tau^{-1/2}\|v\|_{1,\tau} + C_N\tau^{-N}|\gamma_0(v)|_{0,\tau} + \tau^{-3/2}|\gamma_1(v)|_{0,\tau}.\end{aligned} \tag{6.26}$$

for τ sufficiently large and N arbitrary. We can thus estimate the Neumann trace from the Dirichlet trace

$$|\gamma_1(w)|_{0,\tau} \lesssim |\gamma_0(w)|_{1,\tau} + |g_4|_{0,\tau}. \tag{6.27}$$

Now we use the relation (6.2) between the two traces at the boundary, that is, $\gamma_1(w) = K\gamma_0(w) + w_0$. From (6.25), we have

$$\mathrm{Op}_T(l_1) \circ \mathrm{Op}_T(1-a_0)\gamma_0(w) - K\gamma_0(w) = w_0 + g_4. \tag{6.28}$$

(6.28) reads

$$H\gamma_0(w) = w_0 + g_4, \tag{6.29}$$

where $H = \mathrm{Op}_T(l_1(1-a)) - K \mod \Psi^0_{T,\tau}$ and the principal symbol of H in the region where $\tilde{\chi}$ is equal to one is given by

$$h = i(\delta s(x', \xi') - \delta s(x', \tau \mathrm{d}_{x'}\varphi) - \delta\kappa\sigma^2 + \tau\partial_{x_n}\varphi) - 2\delta\tilde{s}(x', \xi', \tau\mathrm{d}_{x'}\varphi) + \rho^+.$$

In order to produce an estimate that is uniform in δ, and handle properly the calculus with the large parameter τ, we write δ as the inverse of a large parameter $\delta = \frac{1}{r}$, $r \ge 1$ and we introduce a new symbolic calculus. We define the order function $M^2 = \lambda^2_{T,\tau} + r\lambda_{T,\tau}$, associated with the usual semi-classical metric on the cotangent bundle $T^*\mathbb{R}^{n-1}$: $g = |\mathrm{d}x'|^2 + \frac{|\mathrm{d}\xi'|^2}{\lambda^2_{T,\tau}}$. The following lemma state that symbols can be defined with this order function, viewing the semi-classical calculus in the general Weyl-Hörmander calculus [12, Section 18.4-6] and [22].

Lemma 6.3. *The order function M^2 is admissible with respect to the metric g, i.e is slowly varying and temperate.*

A proof can be found in [15], Lemma 4.7, in the semi-classical small parameter setting. Actually, any order function defined by a linear combination of powers of $\lambda_{T,\tau}$ is admissible with respect to the metric g. In this symbol classes, we have $rh \in \mathcal{S}(M^2, g)$. The aim is now to construct a parametrix of rh. We have

$$rh = i(s(x', \xi') - s(x', \tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2) + ir\tau\partial_{x_n}\varphi - 2\tilde{s}(x', \xi', \mathrm{d}_{x'}\varphi) + r\rho^+.$$

Taking the imaginary part, $\mathrm{Im}(rh) = s(x', \xi') - s(x', \tau\mathrm{d}_{x'}\varphi) - \kappa\sigma^2 + r\tau\partial_{x_n}\varphi + r\,\mathrm{Im}(\rho^+)$, and remarking that from Lemma 5.3, we have $\mathrm{Re}(s_{\varphi,\sigma}) = s(x', \xi') - s(x', \tau\mathrm{d}_{x'}\varphi) - \kappa\sigma^2 \gtrsim \lambda_{T,\tau}^2$, and that from Lemma 5.4 $r\,\mathrm{Im}(\rho^+) \gtrsim r\lambda_{T,\tau}^1$, we thus obtain $|rh| \gtrsim \mathrm{Im}\, rh \gtrsim \lambda_{T,\tau}^2 + r\lambda_{T,\tau} = M^2$.

Then rh is an elliptic symbol in the class $\mathcal{S}(M^2, g)$, this allows us to construct a parametrix $L \in \mathcal{S}(M^{-2}, g)$ satisfying $\mathrm{Op}_T(L)rH = \mathrm{Op}(\chi_L) + R_L$ with $R_L \in \mathcal{S}_{T,\tau}^{-\infty}$, for some $\chi_L \in \mathcal{S}_{T,\tau}^0$ equal to 1 in a neighborhood of χ and such that $\tilde{\chi}$ equal to 1 in a neighborhood of χ_L. Applying this parametrix to (6.29), we obtain

$$\gamma_0(w) = r\,\mathrm{Op}_T(L)(w_0 + g_4) - R_L\gamma_0(w) + (1 - \mathrm{Op}(\chi_L))\,\mathrm{Op}(\chi_{|_{x_n=0}})\gamma_0(v) \tag{6.30}$$

Yet, we use the following lemma

Lemma 6.4. *For all $u \in \mathscr{S}(\mathbb{R}^{n-1})$ we have*

$$|\,\mathrm{Op}_T(r\lambda_{T,\tau})\,\mathrm{Op}_T(L)u|_{0,\tau} \lesssim |\,\mathrm{Op}_T(\frac{r}{\lambda_{T,\tau} + r})u|_0. \tag{6.31}$$

Proof. There exists $\tilde{u} \in \mathscr{S}(\mathbb{R}^{n-1})$ such that $u = \mathrm{Op}_T(\frac{\lambda_{T,\tau}+r}{r})\tilde{u}$ and it is given by $\mathrm{Op}(\frac{r}{\lambda_{T,\tau}+r})u$. As $r\lambda_{T,\tau} \in \mathcal{S}(r\lambda_{T,\tau}, g)$, $L \in \mathcal{S}(M^{-2}, g)$ and $\lambda_{T,\tau} + r \in \mathcal{S}(\lambda_{T,\tau} + r, g)$ we have $r\lambda_{T,\tau}\#L\#\frac{\lambda_{T,\tau}+r}{r} \in \mathcal{S}(1, g)$ by Theorem 18.5.4 in [12] (stated for the Weyl quantization). Then applying Theorem 18.6.3 in [12], we obtain

$$|\,\mathrm{Op}_T(r\lambda_{T,\tau})\,\mathrm{Op}_T(L)\,\mathrm{Op}_T(\frac{\lambda_{T,\tau} + r}{r})\tilde{u}|_{0,\tau} \lesssim |\tilde{u}|_{0,\tau},$$

that is precisely the result. □

Then using this lemma

$$\begin{aligned} \left|r\,\mathrm{Op}_T(\lambda_{T,\tau})\,\mathrm{Op}_T(L)(w_0 + g_4)\right|_{0,\tau} &\lesssim \left|r\,\mathrm{Op}_T\left((\lambda_{T,\tau}^1 + r)^{-1}\right)(w_0 + g_4)\right|_{0,\tau} \\ &\lesssim \frac{r}{\tau + r}\,|w_0 + g_4|_{0,\tau} \end{aligned} \tag{6.32}$$

and, with (6.3) and (6.26)

$$|w_0 + g_4|_{0,\tau} \lesssim \tau^{-1/2}\|g\|_{0,\tau} + \tau^{-1/2}\|v\|_{1,\tau} + r^{-1}|\gamma_0(v)|_{1,\tau} + |\gamma_0(v)|_{0,\tau} + |\Theta|_{0,\tau} + \tau^{-3/2}|\gamma_1(v)|_{0,\tau}, \tag{6.33}$$

for τ sufficiently large. Observe that in (6.30), $(1 - \mathrm{Op}_T(\chi_L))\,\mathrm{Op}_T(\chi) \in \mathcal{S}_{T,\tau}^{-\infty}$ and $R_L \in \mathcal{S}_{T,\tau}^{-\infty}$ and thus, using (6.30), (6.32) and (6.33), we obtain

$$\frac{r + \tau}{r}\tau^{1/2}|\gamma_0(w)|_{1,\tau} \lesssim \|g\|_{0,\tau} + \|v\|_{1,\tau} + \frac{\tau^{1/2}}{r}|\gamma_0(v)|_{1,\tau} + \tau^{1/2}|\gamma_0(v)|_{0,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{-1/2}|\gamma_1(v)|_{0,\tau}. \tag{6.34}$$

From (6.22), and using (6.7) we have

$$\tau\|w\|_{1,\tau} \lesssim \|g\|_{L^2} + \|v\|_{1,\tau} + \tau^{1/2}|\gamma_0(w)|_{1,\tau} + \tau^{1/2}|\gamma_1(w)|_{0,\tau} + \tau^1|\gamma_1(v)|_{0,\tau} \tag{6.35}$$

Injecting estimates (6.27) and (6.34) in (6.22) yields

$$\begin{aligned} \tau\|w\|_{1,\tau} + \frac{\tau^{1/2}(\tau + r)}{r}|\gamma_0(w)|_{1,\tau} + \tau^{1/2}|\gamma_1(w)|_{0,\tau} \lesssim \|g\|_{0,\tau} & \\ + \|v\|_{1,\tau} + \tau^{-1/2}|\gamma_1(v)|_{0,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{-1/2}|\gamma_0(v)|_{1,\tau}, & \end{aligned} \tag{6.36}$$

by taking τ_0 sufficiently large. Writing $\delta = \frac{1}{r}$ ends the proof. □

6.2 Estimate in $\mathcal{E}^0$

We shall derive an estimate in the region where $P_{\varphi,\sigma}$ is not elliptic. This is precisely the region where there is a loss of a half derivative. However, the operator at the boundary $S_{\varphi,\sigma}$ is elliptic here, this allow us to estimate the two traces from a single observation at the boundary.

Proposition 6.5. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and let $\nu_0 > 0$, $\eta_1 > 0$ be chosen sufficiently small. Let φ be a weight function satisfying the conditions of Section 4.2 in $\overline{V}$, and $\chi_0(x,\xi',\tau,\sigma) \in \mathcal{S}^0_T$ be such that* $\operatorname{supp}(\chi) \subset \mathcal{E}^0$. *Assume moreover that $\partial_{x_n}\varphi \geq C' > 0$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\tau\| \operatorname{Op}_T(\chi_0)v\|^2_{1,\tau} + \delta^2\tau|\operatorname{Op}_T(\chi_0)v_{|x_n=0}|^2_{2,\tau} + \tau|\operatorname{Op}_T(\chi_0)v_{|x_n=0}|^2_{1,\tau} + \tau|D_n\operatorname{Op}(\chi_0)v_{|x_n=0}|^2_{0,\tau}$$
$$\leq C\left(\|P_{\varphi,\sigma}v\|^2_{0,\tau} + \|v\|^2_{1,\tau} + \delta^2\tau|v_{|x_n=0}|^2_{1,\tau} + \tau|v_{|x_n=0}|^2_0 + \tau|\Theta|^2_{0,\tau}\right). \quad (6.37)$$

for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0|\sigma|$, for all $u \in \overline{C}^\infty_0(V^+)$ and $\delta \in (0,1]$.

Remark 6.6. *Observe that $\mathcal{E}^0$ depends on η_1 and on φ and this is precisely the region where theses parameters will be fixed (see Lemma 6.7). Observe also the critical power of τ in the right hand side of the estimate in front of the norm $|v|_{1,\tau}$. This term will not be absorbed directly when we will try to patch the three different estimates. However, this critical term vanishes in the singular limit $\delta = 0$.*

Proof. We set $w = \operatorname{Op}_T(\chi_0)v$. We recall that $\gamma_0(w) = w_{|x_n=0}$ and $\gamma_1(w) = D_n w_{|x_n=0}$. Using Proposition 4.3, we have

$$\tau\|w\|^2_{1,\tau} + \tau\operatorname{Re}\mathcal{B}(w) \lesssim \|P_{\varphi,\sigma}w\|^2_{0,\tau} \quad (6.38)$$

where B is the following quadratic form on the boundary:

$$\mathcal{B}(f) = 2\big(\partial_{x_n}\varphi\gamma_1(f), \gamma_1(f)\big) + \big(A_1\gamma_0(f), \gamma_1(f)\big) + \big(\gamma_1(f), A'_1\gamma_0(f)\big) + \big(A_2\gamma_0(f), \gamma_0(f)\big), \quad (6.39)$$

and $(.,.)$ denotes the scalar product of $L^2(\mathbb{R}^{n-1})$, and the differential operators A_1, A'_1 and A_2 are defined by (4.16) and (4.17). Arguing in the same way as at the beginning of the proof of the estimate in $\mathcal{E}^+$ in Proposition 6.1, and using (4.12), we have

$$\gamma_1(w) = (\operatorname{Op}_T(\chi_0)D_n v)_{|x_n=0} + ([\operatorname{Op}_T(\chi_0), D_n]v)_{|x_n=0} = \operatorname{Op}_T(\chi_0)(Kv_{|x_n=0} + \Theta) + ([\operatorname{Op}_T(\chi_0), D_n]v)_{|x_n=0} = K\gamma_0(w) + G_1.$$

with estimate of the remainder term

$$|G_1|_{0,\tau} \lesssim \delta|\gamma_0(v)|_{1,\tau} + |\gamma_0(v)|_0 + |\Theta|_0, \quad (6.40)$$

as $K \in \delta\mathcal{D}^2_{T,\tau} + \tau\mathcal{D}^0$ (see (4.13)). It follows that

$$\mathcal{B}(w) = \big(2\partial_{x_n}\varphi K\gamma_0(w), K\gamma_0(w)\big) + 4\operatorname{Re}\big(\partial_{x_n}\varphi K\gamma_0(w), G_1\big) + \big(\partial_{x_n}\varphi G_1, G_1\big) + \big(A_1\gamma_0(w), K\gamma_0(w)\big)$$
$$+ \big(K\gamma_0(w), A'_1\gamma_0(w)\big) + \big(A_1\gamma_0(w), G_1\big) + \big(G_1, A'_1\gamma_0(w)\big) + \big(A_2\gamma_0(w), \gamma_0(w)\big). \quad (6.41)$$

By the Cauchy-Schwarz inequality, for the terms involving G_1, we have

$$\Big|4\operatorname{Re}\big(\partial_{x_n}\varphi K\gamma_0(w), G_1\big) + \big(\partial_{x_n}\varphi G_1, G_1\big) + \big(A_1\gamma_0(w), G_1\big) + \big(G_1, A'_1\gamma_0(w)\big)\Big|$$
$$\lesssim |K\gamma_0(w)|_{0,\tau}|G_1|_{0,\tau} + |\gamma_0(w)|_{1,\tau}|G_1|_{0,\tau} + |G_1|^2_{0,\tau}. \quad (6.42)$$

By symbolic calculus, we can write the "principal" terms of (6.41) in the following way

$$2\big(\partial_{x_n}\varphi K\gamma_0(w), K\gamma_0(w)\big) + \big(A_1\gamma_0(w), K\gamma_0(w)\big) + \big(Kw, A'_1\gamma_0(w)\big) + \big(A_2\gamma_0(w), \gamma_0(w)\big)$$
$$= \big((\delta^2 B_4 + \delta B_3 + B_2)\gamma_0(w), \gamma_0(w)\big), \quad (6.43)$$

where the principal symbols of B_4, B_3 and B_2 are respectively

$$b_4 = 2\partial_{x_n}\varphi_{|x_n=0}|s_{\varphi,\sigma}|^2 \in \mathcal{S}^4_{T,\tau},$$
$$b_3 = 4\tau(\partial_{x_n}\varphi_{|x_n=0})^2 s_2(x,\xi',\tau,\sigma) + 4s_1(x,\xi',\tau)\tilde{r}(x', x_n = 0, \xi', \mathrm{d}_{x'}\varphi_{|x_n=0}) \in \mathcal{S}^3_{T,\tau},$$
$$b_2 = 2\left(\partial_{x_n}\varphi^3\tau^2 + 2\partial_{x_n}\varphi(-r(x,\xi') + p(x,\tau\mathrm{d}_x\varphi) + \sigma^2)\right)_{|x_n=0} \in \mathcal{S}^2_{T,\tau}.$$

Now we state positivity result of these symbols.

Lemma 6.7. *There exists $C > 0$, $\eta_1 > 0$, satisfying $\eta_1 < C_0$, independent of σ, τ and δ such that*

$$b_j(x', \xi', \tau, \sigma) \geq C\lambda_{T,\tau}^j, \qquad j = 2, 3, 4, \tag{6.44}$$

for all (x, ξ', τ, σ) satisfying $-2\eta_1\tau^2 \leq \mu(x, \xi', \tau, \sigma) \leq 2\eta_1\tau^2$, where C_0 is the constant given in the second part of Proposition 5.3.

Proof of Lemma 6.7. The positivity of the symbol b_4 comes precisely from the fact that the boundary operator $S_{\varphi,\sigma}$ is elliptic in this region (see Lemma 5.3). Let us prove now the positivity of b_2. Let $(x', x_n = 0, \xi', \tau, \sigma) \in \mathcal{E}^0$, i.e be such that $-2\eta_1\lambda_{T,\tau}^2 \leq -\mu \leq 2\eta_1\lambda_{T,\tau}^2$. In particular, we have

$$-2\eta_1\lambda_{T,\tau}^2 \leq \sigma^2 + p(x', x_n = 0, \tau \mathrm{d}_{x'}\varphi) - r(x', x_n = 0, \xi'),$$

and thus

$$b_2 \geq 2\partial_{x_n}\varphi^3\tau^2 - 4\eta_1\partial_{x_n}\varphi\lambda_{T,\tau}^2. \tag{6.45}$$

Observe that as $(x', x_n = 0, \xi', \tau, \sigma) \in \mathcal{E}^0$, there exists $C > 0$ such that $\lambda_{T,\tau}^2 \leq C\tau^2$ (see Lemma 5.2) for η_1 sufficiently small. In addition, if η_1 is chosen sufficiently small, from (6.45) we have the positivity of b_2.
Let us finally prove the positivity of b_3. From Lemma 5.3, we have for η_1 sufficiently small

$$s_2(x', \xi', \tau, \sigma) = s(x', \xi') - s(x', \tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2 \gtrsim \lambda_{T,\tau}^2.$$

Moreover,

$$4|s_1(x, \xi', \tau)\tilde{r}(x', x_n = 0, \xi', \mathrm{d}_{x'}\varphi)| \lesssim \tau|\xi'|^2|\mathrm{d}_{x'}\varphi|^2 \lesssim \nu_0^2\tau|\xi'|^2|\partial_{x_n}\varphi|^2, \tag{6.46}$$

since we have (4.15), and again for ν_0 sufficiently small, we obtain $b_3 \gtrsim \tau\lambda_{T,\tau}^2$ and using again that $\lambda_{T,\tau} \lesssim \tau$ in the region $\mathcal{E}^0$, we conclude the proof. $\square$
We can now apply the microlocal Gårding inequality in (6.43), and taking τ sufficiently large we obtain, for an arbitrary $N \in \mathbb{N}$

$$\begin{aligned}\mathcal{B}(w) \geq C(\delta^2|\gamma_0(w)|_{2,\tau}^2 + \delta|\gamma_0(w)|_{3/2,\tau}^2 + |\gamma_0(w)|_{1,\tau}^2) \\ - C'\left(|K\gamma_0(w)|_{0,\tau}|G_1|_{0,\tau} + |\gamma_0(w)|_{1,\tau}|G_1|_{0,\tau} + |G_1|_{0,\tau}^2\right) - C_N|\gamma_0(v)|_{-N,\tau}^2.\end{aligned} \tag{6.47}$$

By the Young inequality, the right hand side of (6.47) reads

$$|K\gamma_0(w)|_{0,\tau}|G_1|_{0,\tau} + |\gamma_0(w)|_{1,\tau}|G_1|_{0,\tau} + |G_1|_{0,\tau}^2 \lesssim \delta'(\delta^2|\gamma_0(w)|_{2,\tau}^2 + \tau^2|\gamma_0(w)|_{0,\tau}^2 + |\gamma_0(w)|_{1,\tau}^2) + \delta'^{-1}|G_1|_{0,\tau}^2, \tag{6.48}$$

for all $\delta' > 0$. From (6.38), (6.47) and (6.48), we obtain

$$\tau\|w\|_{1,\tau}^2 + \tau\left(\delta^2|\gamma_0(w)|_{2,\tau}^2 + |\gamma_0(w)|_{1,\tau}^2\right) \lesssim \|P_{\varphi,\sigma}w\|_{0,\tau}^2 + \tau|G_1|_{0,\tau}^2 + C_N\tau|\gamma_0(v)|_{-N,\tau}^2.$$

Using the estimate of $|G_1|_{0,\tau}$ in (6.40) we obtain the sought result. $\square$

6.3 Estimate in $\mathcal{E}^-$

In this region, we have $\mu < -\eta_1$ implying $|\xi'| \lesssim \tau$, and the operator at the boundary $S_{\varphi,\sigma}$ is not elliptic. However, in the case where $\partial_{x_n}\varphi > 0$, the two roots of $p_{\varphi,\sigma}$ are of negative imaginary parts, and we can estimate the two traces at the boundary with no observation term.

Proposition 6.8. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and φ be a weight function satisfying the conditions of Section 4.2 in $\overline{V}$, and $\chi(x, \xi', \tau, \sigma) \in \mathcal{S}_T^0$ be such that $\mathrm{supp}(\chi) \subset \mathcal{E}^-$. Assume moreover that $\partial_{x_n}\varphi \geq C' > 0$ on $\overline{V}$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\|\operatorname{Op}_T(\chi)v\|_{2,\tau}^2 + \tau|\operatorname{Op}_T(\chi)v_{|x_n=0}|_{1,\tau}^2 + \tau|D_n\operatorname{Op}_T(\chi)v_{|x_n=0}|_{0,\tau}^2 \leq C\left(\|P_{\varphi,\sigma}v\|_{0,\tau}^2 + \|v\|_{1,\tau}^2\right),$$

for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0|\sigma|$, $u \in \overline{C_0^\infty}(V^+)$ and $\delta \in (0, 1]$.

Proof. We follow the proof of the microlocal estimate in the $\mathcal{E}^+$ region to (6.12) and (6.13). We remark that the integral along the contour β is identically 0, because the integrand is holomorphic and the two poles have a non-positive imaginary part. We then have

$$\underline{w} = E_M(\underline{P_{\varphi,\sigma} w}) + g_1, \tag{6.49}$$

where g_1 is a remainder coming from microlocalisations. We recall that it can be estimated by (6.9). Thus, (6.49) yields

$$\|w\|_{2,\tau} \lesssim \|P_{\varphi,\sigma} v\|_{L^2} + \|v\|_{1,\tau}.$$

Taking the γ_0-trace on $\{x_n = 0\}$ in (6.49), and thanks to the trace inequality (3.2)

$$\tau^{1/2}|w|_{1,\tau} \lesssim \|P_{\varphi,\sigma} v\|_{L^2} + \|v\|_{1,\tau},$$

and finally taking the γ_1-trace

$$\tau^{1/2}|D_n w|_0 \lesssim \|P_{\varphi,\sigma} v\|_{L^2} + \|v\|_{1,\tau}.$$

Patching these three estimates, we obtain the sought result. □

7 Proof of the local Carleman estimates

7.1 A local Carleman estimate from the interior up to the boundary

From the three microlocal estimates of the previous section, we shall derive a local Carleman estimate with observation from the interior, that is, with the condition $\partial_{x_n}\varphi > 0$. With this sign condition, from Propositions 6.1, 6.5 and 6.8, we recall that we have the three following estimates:

$$\| \operatorname{Op}_T(\chi_-)v\|_{2,\tau} + \tau^{1/2}| \operatorname{Op}_T(\chi_-)v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|D_n \operatorname{Op}_T(\chi_-)v_{|x_n=0}|_{0,\tau} \lesssim \|P_{\varphi,\sigma} v\|_{0,\tau} + \|v\|_{1,\tau}, \tag{7.1}$$

$$\begin{aligned}\tau^{1/2}\| \operatorname{Op}_T(\chi_0)v\|_{1,\tau} + \delta\tau^{1/2}| \operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{2,\tau} + \tau^{1/2}\left(| \operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{1,\tau} + |D_n \operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{0,\tau}\right)\\ \lesssim \|P_{\varphi,\sigma} v\|_{0,\tau} + \|v\|_{1,\tau} + \delta\tau^{1/2}|v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|v_{|x_n=0}|_{0,\tau} + \tau^{1/2}|\Theta|_{0,\tau},\end{aligned} \tag{7.2}$$

$$\begin{aligned}\tau\| \operatorname{Op}_T(\chi_+)v\|_{1,\tau} + (\tau^{1/2} + \delta\tau^{3/2})| \operatorname{Op}_T(\chi_+)v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|D_n \operatorname{Op}_T(\chi_+)v_{|x_n=0}|_{0,\tau}\\ \lesssim \|P_{\varphi,\sigma} v\|_{0,\tau} + \|v\|_{1,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{-1/2}(|v_{|x_n=0}|_{1,\tau} + |D_n v_{|x_n=0}|_{0,\tau}),\end{aligned} \tag{7.3}$$

where the cut-off functions χ_+, χ_- and χ_0 are respectively supported in $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathcal{E}^0$, and for η_1 chosen sufficiently small. We recall that $P_{\varphi,\sigma}$ and Θ are defined in Section 4.1. We shall denote respectively equalities (7.1), (7.2) and (7.3) by

$$LHS_-(v) \lesssim RHS_-(v), \quad LHS_0(v) \lesssim RHS_0(v), \quad LHS_+(v) \lesssim RHS_+(v). \tag{7.4}$$

Due to the critical term $\delta\tau^{1/2}|v|_{x_n=0}|_{1,\tau}$ on the right hand side of (7.2) and (7.4), we cannot patch directly these estimates and absorb some of the terms on the right hand side by other terms on the left hand side by chosen parameters appropriately. This is the reason for the introduction of an additionnal small parameter $\varepsilon > 0$, independent of δ, τ and σ. Now, we take a partition of unity: $\chi_- + \chi_0 + \chi_+ = 1$ satisfying $\chi_+, \chi_-, \chi_0 \in \mathcal{S}^0_{T,\tau}$ and moreover

$$\operatorname{supp}(\chi_+) \subset \mathcal{E}^+, \qquad \operatorname{supp}(\chi_0) \subset \mathcal{E}^0, \qquad \operatorname{supp}(\chi_-) \subset \mathcal{E}^-.$$

For the construction of χ_+, χ_- and χ_0, take for instance $\zeta_0 \in C_0^\infty(\mathbb{R})$, such that $\zeta_0 = 1$ on $[-\eta_1, \eta_1]$, $\operatorname{supp}(\zeta_0) \subset (-2\eta_1, 2\eta_1)$ and set

$$\chi_0 = \zeta_0(\frac{\mu}{\lambda^2_{T,\tau}}), \quad \chi_- = \mathbf{1}_{(-\infty,0]}(\mu)(1-\chi_0), \quad \chi_+ = \mathbf{1}_{[0,+\infty)}(\mu)(1-\chi_0).$$

Observe that this construction yields indeed tangential symbols in $\mathcal{S}^0_{T,\tau}$ by adapting [12, Theorem 18.1.10].

We then write, for $0 < \varepsilon \le 1$

$$LHS_-(v) + \varepsilon LHS_0(v) + LHS_+(v) \lesssim LHS_-(v) + \varepsilon RHS_0(v) + LHS_+(v) \tag{7.5}$$

Consider the critical term $\varepsilon\delta\tau^{1/2}|v_{|x_n=0}|_{1,\tau}$ on the right hand side of (7.5). We write

$$\begin{aligned}\varepsilon\delta\tau^{1/2}|v_{|x_n=0}|_{1,\tau} \le & \varepsilon\delta\tau^{1/2}\left(|\operatorname{Op}_T(\chi_-)v_{|x_n=0}|_{1,\tau} + |\operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{1,\tau} + |\operatorname{Op}_T(\chi_+)v_{|x_n=0}|_{1,\tau}\right).\\ \le & \varepsilon LHS_-(v) + \tau^{-1}\varepsilon LHS_0(v) + \varepsilon\tau^{-1}LHS_+(v)\end{aligned}$$

By choosing $\varepsilon = \varepsilon_1 > 0$ sufficiently small and τ sufficiently large, we obtain

$$LHS_-(v) + LHS_0(v) + LHS_+(v) \lesssim \|P_{\varphi,\sigma}v\|_{0,\tau} + \|v\|_{1,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{1/2}|v_{|x_n=0}|_{1,\tau} + |D_n v|_{0,\tau}.$$

We conclude by taking τ sufficiently large, and we obtain the following Carleman estimate in the neighborhood of the boundary

$$\tau^{1/2}\|v\|_{1,\tau} + \tau^{1/2}|v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|D_n v_{|x_n=0}|_{0,\tau} \lesssim \|P_{\varphi,\sigma}v\|_{0,\tau} + \tau^{1/2}|(D_n - K)v_{|x_n=0}|_{0,\tau}.$$

Then coming back to $u = e^{-\tau\varphi}v$ yields

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}Du\|_{L^2} + \tau^{3/2}|e^{\tau\varphi}u_{|x_n=0}|_{L^2} + \tau^{1/2}|e^{\tau\varphi}Du_{|x_n=0}|_{L^2} \lesssim \|e^{\tau\varphi}P_\sigma u\|_{L^2} + \tau^{1/2}|e^{\tau\varphi}(\partial_{x_n} - \delta S_\sigma)u|_{L^2}.$$

This concludes the proof of the second part of Theorem 1.5. As the conditions imposed to the weight function in Section 4.2 are invariant under change of coordinates, we naturally obtain (1.25) in the neighborhood of a point of the boundary $\partial\Omega$.

7.2 A local Carleman estimate with a boundary observation

Here, we prove (1.24), that is, an estimation with a boundary observation. this will be used below to prove Theorem 1.3 in the case $a = 0$ and $b \neq 0$ (see (1.9) and (1.16)). There, b is a non-negative bounded function satisfying $b \ge C > 0$ on an open subset ω_B of $\partial\Omega$. We then have to observe from the boundary and thus the weight function is chosen such that $\partial_\nu\varphi > 0$ in a neighborhood of a point of ω_B. Hence, it is sufficient to prove a Carleman estimate where there are no assumptions about the sign of $\partial_\nu\varphi$, yet assuming $\partial_\nu\varphi \neq 0$. Observing that Proposition 6.1 is independent of the sign of $\partial_\nu\varphi$, it remains to prove an estimate in the region $\mathcal{F} := \mathcal{E}^0 \cup \mathcal{E}^-$.

Proposition 7.1. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and φ be a weight function satisfying the conditions of Section 4.2 in $\overline{V}$ with in particular $|\partial_\nu\varphi| \ge C_0 > 0$, and $\chi \in \mathcal{S}^0_T$ be such that $\operatorname{supp}\chi \subset \mathcal{F}$. Then there exists $C > 0$ and $\tau_0 > 0$ such that*

$$\tau\|\operatorname{Op}_T(\chi)v\|^2_{1,\tau} + \tau|D_n\operatorname{Op}_T(\chi)v_{|x_n=0}|^2_{0,\tau} \le C\left(\|P_{\varphi,\sigma}v\|^2_{0,\tau} + \|v\|^2_{1,\tau} + (\delta^2\tau^5 + \tau^3)|v_{|x_n=0}|^2_{0,\tau} + \tau|\Theta|^2_{0,\tau}\right). \tag{7.6}$$

for all $|\sigma| \ge 1$, $\tau \ge \tau_0|\sigma|$ and $v \in \overline{C}^\infty_0(V^+)$.

Proof.
We set $w := \operatorname{Op}_T(\chi)v$. Using Proposition 4.3, there exists $\tau_0 > 0$ such that

$$\tau\|w\|^2_{1,\tau} + \tau\operatorname{Re}\mathcal{B}(w) \lesssim \|P_{\varphi,\sigma}w\|^2_{0,\tau} \lesssim \|P_{\varphi,\sigma}v\|^2_{0,\tau} + \|v\|^2_{1,\tau}, \tag{7.7}$$

for $\tau \ge \tau_0|\sigma|$, where the boundary form $\mathcal{B}$ reads

$$\begin{aligned}\mathcal{B}(w) = & \left(\partial_{x_n}\varphi D_n w_{|x_n=0}, D_n w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \left(A_1 w_{|x_n=0}, D_n w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}\\ & + \left(D_n w_{|x_n=0}, A_1' w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \left(A_2 w_{|x_n=0}, w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}.\end{aligned}$$

Observe that, by symbolic calculus, for all $N \in \mathbb{N}$,

$$A_j \circ \operatorname{Op}_T(\chi) = \operatorname{Op}_T(c_j) + A_j^N \quad (\text{resp. } A_j' \circ \operatorname{Op}_T(\chi) = \operatorname{Op}_T(c_j') + A_j'^N)$$

with c_j (resp. c_j') $\in \mathcal{S}^j_{T,\tau}$, $\operatorname{supp}(c_j)$ (resp. $\operatorname{supp}(c_j')$) $\subset \operatorname{supp}(\chi)$, A_j^N (resp. $A_j'^N$) $\in \Psi^{-N}_{T,\tau}$. Moreover, as $|\xi'| \lesssim \tau$ in $\operatorname{supp}(\chi)$ by Lemma 5.2, we have $c_j, c_j' \in \tau^j\mathcal{S}^0_{T,\tau}$. Hence, we have the estimate

$$|\mathcal{B}(w)| \lesssim |(D_n\operatorname{Op}_T(\chi)v)_{|x_n=0}|^2_{0,\tau} + \tau^2|v_{|x_n=0}|^2_{0,\tau}. \tag{7.8}$$

Now we shall use the boundary condition (4.12). With the same arguments as above, we have $\mathrm{Op}_T(\chi)K \in (\delta\tau^2 + \tau)\psi^0_{T,\tau}$ and thus

$$\begin{aligned}|(D_n \mathrm{Op}_T(\chi)v)_{|_{x_n=0}}|_{0,\tau} &\leq |(\mathrm{Op}_T(\chi)D_n v)_{|_{x_n=0}}|_{0,\tau} + |([D_n, \mathrm{Op}_T(\chi)]v)_{|_{x_n=0}}|_{0,\tau} \\ &\lesssim |(\mathrm{Op}_T(\chi)Kv)_{|_{x_n=0}}|_{0,\tau} + |([D_n, \mathrm{Op}_T(\chi)]v)_{|_{x_n=0}}|_{0,\tau} + |\Theta|_{0,\tau} \\ &\lesssim (\delta\tau^2 + \tau)|v_{|_{x_n=0}}|_{0,\tau} + |\Theta|_{0,\tau}.\end{aligned}$$

This, in addition of (7.7) and (7.8) yields the result. □

We can now conclude the proof of the first part of Theorem 1.5 by patching estimate (7.6) with estimate (6.1), since no condition on the sign of $\partial_{x_n}\varphi$ is needed in the region $\mathcal{E}^+$, and taking τ sufficiently large to absorb low order terms. Note that, as opposed to the proof of Section 7.1, there is no need for the introduction of the additional small parameter ε.

8 Interpolation and resolvent estimate

8.1 Interpolation estimate

8.1.1 Observation from the interior

In this section, following the ideas in [20], we shall derive an interpolation inequality from the Carleman estimate (1.25). We prove the inequality in the interior, and propagate it up to the boundary. Below, we set $\Omega_\eta = \{x \in \Omega,\ d(x, \partial\Omega) > \eta\}$ and $\Omega^\eta = \{x \in \Omega,\ d(x, \partial\Omega) < \eta\}$.

Theorem 8.1. *Let ω_I be an open subset of Ω. There exists $C > 0$ such that*

$$\|u\|_{\mathcal{V}_\delta} \leq Ce^{C|\sigma|}\left(\|P_\sigma u\|_{L^2(\Omega)} + |(\partial_\nu + \delta S_\sigma)u|_{L^2(\partial\Omega)} + \|u\|_{L^2(\omega_I)}\right),$$

for all $|\sigma| \geq 1$ and $u \in H^2(\Omega)$ such that $u_{|\partial\Omega} \in H^2(\partial\Omega)$. We recall that $\|.\|_{\mathcal{V}_\delta}$ is the norm defined in (1.13).

The proof is splitted in the two following lemmata. First, an interpolation estimate in a neighborhood of the boundary

Lemma 8.2. *Let $y_0 \in \partial\Omega$.Then, there exist $\eta > 0$, a neighborhood V of y_0 in $\overline{\Omega}$, $\mu \in (0, 1]$, and $C > 0$ such that*

$$\|u\|_{H^1(V)} + |u_{|\partial\Omega}|_{H^1(V\cap\partial\Omega)} \leq Ce^{C\sigma}\left(\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu}\left(\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + \|u\|_{H^1(\Omega_\eta)}\right)^\mu$$

for all $u \in H^2(\Omega)$ such that $u_{|\partial\Omega} \in H^2(\partial\Omega)$, for all $|\sigma| \geq 1$, and for all $\delta > 0$.

Second, an interpolation estimate in the interior, which proof can be found in Appendix D.4. A version without the parameter σ can be found in [20].

Lemma 8.3. *Let Ω be a connected open set of $\mathbb{R}^n$ and ω_I be an open set compactly embedded in Ω. Then, for all $\eta > 0$, there exists $C > 0$, and $\mu \in (0, 1]$ such that*

$$\|u\|_{H^1(\Omega_\eta)} \leq Ce^{C|\sigma|}\|u\|^{1-\mu}_{H^1(\Omega)}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega_I)}\right)^\mu$$

for all $u \in H^2(\Omega)$, $|\sigma| \geq 1$.

Proof of Lemma 8.2 We shall work in normal geodesic coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ as in Section 4 in a neighborhood U in $\overline{\Omega}$ of $y_0 \in \partial\Omega$. These coordinates are chosen such that $y_0 = 0$. We shall still denote U the corresponding neighborhood of 0 in $\overline{\mathbb{R}^n_+}$. For $\beta > 0$, we define the following anisotropic distance on $\mathbb{R}^n$:

$$d_\beta(x, y) = \left(|x' - y'|^2 + \beta|x_n - y_n|^2\right)^{1/2}. \tag{8.1}$$

We choose $r_0 > 0$ such that $x_0 = (0, r_0) \in U$. Then we take W a neighborhood of 0 in $\mathbb{R}^n$ such that $W^+ := W \cap \mathbb{R}^n_+ \subset U$ and such that $d(x_0, W^+) > 0$. We set $\varphi = e^{-\lambda\psi}$ where $\psi(x) = d_\beta(x, x_0)$. Observe that φ is an admissible weight function on W^+ for λ sufficiently large (see Proposition 4.2) and for β sufficiently large (see Section 4.2). Until the end of the proof, β is kept fixed. We define the following cut-off functions $\chi_0,\ \chi_1 \in C^\infty_0(\mathbb{R}^n)$:

$$\chi_0(x) = \begin{cases} 0 & \text{in} \quad x_n > r_1 \\ 1 & \text{in} \quad x_n \in [0, r_1/2] \end{cases} \quad ; \quad \chi_1(x) = \begin{cases} 0 & \text{if} \quad d_\beta(x, x_0) \leq r_2 \text{ or } d_\beta(x, x_0) > r_5 \\ 1 & \text{in} \quad d_\beta(x, x_0) \in [r_3, r_4], \end{cases}$$

with $0 < r_1 < r_0$ and $0 < r_2 < r_3 < r_4 < r_5$ such that

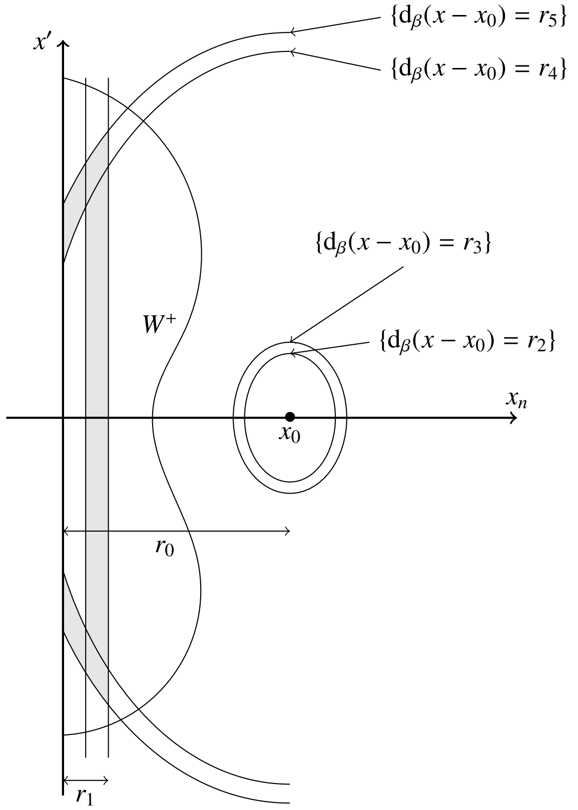


Figure 4: Geometry of the cut-off functions near $(0,0)$. The grey zone corresponds to the region where $\chi_0\chi_1$ varies.

- r_3 is small enough to have $\overline{B_\beta(x_0,r_3)} \cap \overline{W^+} = \emptyset$, where B_β denotes the open ball associated with the distance d_β;
- r_1 small enough and $r_0 < r_4 < r_5$ are such that $\{x \in \mathbb{R}^n_+ \mid x_n \leq r_1\} \cap \{r_4 \leq d_\beta(x,x_0) \leq r_5\} \subset W^+$.

The geometry of the supports of the cut-off functions is represented in Figure 4. As $\partial_{x_n}\varphi \geq C > 0$ on $\operatorname{supp}\chi_0\chi_1$, we can apply the Carleman estimate (1.25) on W^+ to $w = \chi_0\chi_1 u$: there exist $\tau_0 > 0$ such that

$$\tau^3\|e^{\tau\varphi}w\|^2_{L^2(W^+)} + \tau\|e^{\tau\varphi}Dw\|^2_{L^2(W^+)} + \tau^3|e^{\tau\varphi}w_{|x_n=0}|^2_{L^2(W^0)} + \tau|e^{\tau\varphi}D'w_{|x_n=0}|^2_{L^2(W^0)} \lesssim \|e^{\tau\varphi}P_\sigma w\|^2_{L^2(W^+)} + \tau|(e^{\tau\varphi}(\partial_{x_n} - \delta S_\sigma)w)_{|x_n=0}|^2_{L^2(W^0)}, \tag{8.2}$$

for all $\tau \geq \tau_0|\sigma|$, where $W^0 = W \cap \{x_n = 0\}$.The right hand side can be estimated as follows

$$\begin{aligned}\|e^{\tau\varphi}P_\sigma w\|_{L^2(W^+)} &\lesssim \|e^{\tau\varphi}P_\sigma u\|_{L^2(W^+)} + \|e^{\tau\varphi}[P_\sigma,\chi_0\chi_1]u\|_{L^2(W^+)}\\ &\lesssim e^{\tau C_3}\|P_\sigma u\|_{L^2(W^+)} + e^{\tau C_3}\|u\|_{H^1(W^+\cap\{x_n\in[r_1/2,r_1]\})} + e^{\tau C_1}\|u\|_{H^1(W^+)},\end{aligned}$$

with $C_3 > e^{-\lambda(r_0-r_1)}$ and $C_1 = e^{-\lambda r_4}$. Observe that $C_1 < C_3$. Here, we used that the weight function φ is radial with respect to the distance d_β to x_0 and decreasing as x moves away from x_0, and the commutator $[P_\sigma,\chi_0\chi_1]$ is a differential operator of order 1 supported in the region were $\chi_0\chi_1$ varies (represented in grey in Figure 4).
In the same spirit, using that $\delta \leq 1$

$$|(e^{\tau\varphi}(\partial_{x_n} - \delta S_\sigma)w)_{|x_n=0}|_{L^2(W^0)} \lesssim e^{\tau C_3}|(\partial_{x_n} - \delta S_\sigma)u_{|x_n=0}|_{L^2(W^0)} + e^{\tau C_1}\left(|u_{|x_n=0}|_{L^2(W^0)} + |D'u_{|x_n=0}|_{L^2(W^0)}\right).$$

Finally, we can restrict the left hand side of the Carleman estimate to $\tilde{W} := B(0,r_6) \cap \{x_n > 0\}$ with $r_6 > 0$ taken sufficiently small to have $\chi_0\chi_1 = 1$ on $B(0,r_6)$ and this yields, for $\tau \geq 1$,

$$\tau^{3/2}\|e^{\tau\varphi}\chi_0\chi_1 u\|_{L^2(\tilde{W})} + \tau^{1/2}\|e^{\tau\varphi}D\chi_0\chi_1 u\|_{L^2(\tilde{W})} + \tau^{3/2}|e^{\tau\varphi}\chi_0\chi_1 u_{|x_n=0}|_{L^2(\partial\tilde{W}\cap\{x_n=0\})} + \tau^{1/2}|e^{\tau\varphi}D'\chi_0\chi_1 u_{|x_n=0}|_{L^2(\partial\tilde{W}\cap\{x_n=0\})} \gtrsim e^{\tau C_2}(\|u\|_{H^1(\tilde{W})} + |u_{|x_n=0}|_{H^1(\partial\tilde{W}\cap\{x_n=0\})}), \tag{8.3}$$

where $C_2 = \inf_{\overline{W}} \varphi$. We finally obtain, coming back to the original coordinates, for some $\eta > 0$,

$$\|u\|_{H^1(V)} + |u_{|\partial\Omega}|_{H^1(V\cap\partial\Omega)} \lesssim e^{\tau(C_3-C_2)} \left(\|P_\sigma u\|_{L^2(\Omega)} + |(\partial_\nu + \delta S_\sigma)u_{|\partial\Omega}|_{L^2(\partial\Omega)} + \|u\|_{H^1(\Omega_\eta)}\right) + e^{-\tau(C_2-C_1)} \left(\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right), \quad (8.4)$$

where V is an open neighborhood of $y_0 \in \partial\Omega$ in $\overline{\Omega}$. Note that we have $0 < C_1 < C_2 < C_3$. Optimizing this inequality by applying the following lemma, which proof is given in Appendix D.5 concludes the proof.

Lemma 8.4. *Let $A, B, C \geq 0$ such that $A \leq C$. Suppose there exists $\tilde{\tau}, \beta, \gamma > 0$ such that $A \leq e^{\beta\tau} B + e^{-\gamma\tau} C$ for all $\tau \geq \tilde{\tau}$. Then*

$$A \leq K B^\mu C^{1-\mu},$$

with $K = \max((\gamma/\beta)^{\frac{\beta}{\beta+\gamma}} + (\beta/\gamma)^{\frac{\gamma}{\beta+\gamma}}, (\beta/\gamma)^{\frac{\gamma}{\beta+\gamma}} e^{\gamma\tilde{\tau}})$ and $\mu = \frac{\gamma}{\beta+\gamma}$.

□

We end this section by the proof of the theorem.

Proof of Theorem 8.1

We shall in fact prove the stronger estimate

$$\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \lesssim e^{C|\sigma|} \left(\|P_{\varphi,\sigma} u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + \|u\|_{L^2(\omega_I)}\right). \quad (8.5)$$

As $\delta \leq 1$, the result follows. Observe that we can assume that u satisfies

$$\begin{cases} \|P_\sigma u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \\ |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \leq \|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}, \end{cases} \quad (8.6)$$

otherwise (8.5) follows immediatly. From Lemma 8.2 and by a compactness argument, we can find $\eta' > 0$, $\eta'' > 0$, $C > 0$ and $\mu' > 0$ such that

$$\|u\|_{H^1(\Omega^{\eta'})} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \lesssim e^{C|\sigma|} \left(\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu'} (\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + \|u\|_{H^1(\Omega_\eta)})^{\mu'}, \quad (8.7)$$

for all $|\sigma| \geq 1$ and $\delta \in (0, 1]$ and all $0 < \eta \leq \eta''$ (observing that Ω_η increases as η decreases). By Lemma 8.3, for all $\eta > 0$, there exists $C' > 0$ such that

$$\|u\|_{H^1(\Omega_\eta)} \lesssim e^{C'|\sigma|} \|u\|_{H^1(\Omega)}^{1-\mu} \left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega_I)}\right)^\mu, \quad (8.8)$$

for all $|\sigma| \geq 1$ and $\delta \in (0, 1]$. Using (8.6) and (8.8), we have

$$\|u\|_{H^1(\Omega_\eta)} + \|P_\sigma u\|_{L^2(\Omega)} + |\partial_n u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \lesssim e^{C'|\sigma|} \left(\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu} \times \left(\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + \|u\|_{L^2(\omega_I)}\right)^\mu. \quad (8.9)$$

Injecting estimate (8.9) in (8.7), we obtain (8.5). □

8.1.2 Observation from the boundary

We shall prove here the following theorem (which is the counterpart of Theorem 8.1 in the boundary case).

Theorem 8.5. *Let $\omega_B \subset \partial\Omega$ be an open subset of the boundary. Then there exists $C > 0$ such that*

$$\|u\|_{\mathcal{V}_\delta} \leq C e^{C|\sigma|} \left(\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + |u_{|\partial\Omega}|_{L^2(\omega_B)}\right),$$

for all $|\sigma| \geq 1$, $u \in H^2(\Omega)$ such that $u_{|\partial\Omega} \in H^2(\partial\Omega)$.

Proof. Observe first that we can assume

$$\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \leq \|u\|_{\mathcal{V}_\delta(\Omega)}, \tag{8.10}$$

otherwise, the estimate is immediate. We use normal geodesic coordinates (x', x_n) in a neighborhood V of a point $y_0 \in \omega_B$ such that $V \subset \omega_B$. Hence, we consider an open neighborhood W of $(0,0)$ in $\mathbb{R}^n$ such that $W \cap \{x_n = 0\} \subset V$, and we set $\varphi = e^{\lambda\psi}$, where $\psi(x) = -\mathrm{d}_\beta(x, x_0)$, $x_0 = (0, -r_0)$, $r_0 > 0$ will be fixed below and d_β denotes the anisotropic distance defined in (8.1). Observe that for λ sufficiently large and β sufficiently small, φ fulfills the weight function properties required in Section 4.2 in $\overline{W \cap \mathbb{R}^n_+}$. We now define the following C_0^∞ cut-off function

$$\chi(x) = \begin{cases} 1 & \text{if} \quad \mathrm{d}_\beta(x, x_0) < r_1 \\ 0 & \text{if} \quad \mathrm{d}_\beta(x, x_0) > r_2, \end{cases}$$

where the r_j are such that $r_0 < r_1 < r_2$ and $\{x \in \mathbb{R}^n,\ x_n \geq 0\,, \mathrm{d}_\beta(x, x_0) \leq r_2\} \subset \overline{W^+}$. We recall that $W^+ = W \cap \{x_n > 0\}$ and $W^0 = W \cap \{x_n = 0\}$. We can then apply the Carleman estimate (1.24) to χu in W^+, as in the present case $\partial_\nu\varphi \geq C > 0$. We have

$$\tau^3\|e^{\tau\varphi}\chi u\|^2_{L^2(W^+)} + \tau\|e^{\tau\varphi}\nabla\chi u\|^2_{L^2(W^+)} \lesssim \|e^{\tau\varphi}P_\sigma\chi u\|^2_{L^2(W^+)} + \tau|(\partial_{x_n} - \delta S_\sigma)\chi u_{|x_n=0}|_{L^2(W^0)} + \tau^5|e^{\tau\varphi}\chi u_{|x_n=0}|_{L^2(W^0)}.$$

The right hand side can be estimated as follows (using the fact that commutators are supported in regions where χ varies)

$$\begin{aligned}\|e^{\tau\varphi}P_\sigma(\chi u)\|^2_{L^2(W^+)} + \tau|(\partial_{x_n} - \delta S_\sigma)\chi u_{|x_n=0}|_{L^2(W^0)} + \tau^5|e^{\tau\varphi}\chi u_{|x_n=0}|_{L^2(W^0)} \lesssim e^{\tau C_3}\Big(\|P_\sigma u\|_{L^2(W^+)} \\ + |(\partial_{x_n} - \delta S_\sigma)u_{|x_n=0}|_{L^2(W^0)} + |u_{|x_n=0}|_{L^2(W^0)}\Big) + e^{\tau C_1}\left(\|u\|_{H^1(W^+)} + |u_{|x_n=0}|_{H^1(W^0)}\right).\end{aligned}$$

with $C_3 > \varphi(0, r_0)$ and $C_1 = \varphi(0, r_1)$. We can restrict the left hand side to $\tilde{W}$, an open subset compactly embedded in $W^+ \cap \{x \in \mathbb{R}^n,\ 0 < \mathrm{d}_\beta(x_{0,x}) < r_1\}$, to obtain

$$\begin{aligned}\|u\|_{H^1(\tilde{W})} \leq e^{\tau(C_3 - C_2)}\left(\|P_\sigma u\|_{L^2(W^+)} + |(\partial_{x_n} - \delta S_\sigma)u_{|x_n=0}|_{L^2(\omega_B)} + |u_{|x_n=0}|_{L^2(w_B)}\right) \\ + e^{-\tau(C_2 - C_1)}\left(\|u\|_{H^1(W^+)} + |u_{|x_n=0}|_{H^1(\omega_B)}\right).\end{aligned}$$

where $C_2 = e^{\lambda \mathrm{d}_\beta(x_0, \tilde{W})}$. Observe that $C_1 < C_2 < C_3$. Coming back to the original coordinates in the neighborhood of y_0, and optimizing this estimate using Lemma 8.4, we obtain that there exist $1 \geq \mu > 0$ and $C > 0$ such that

$$\|u\|_{L^2(O)} \leq Ce^{C|\sigma|}(\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)})^{1-\mu}\left(\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + |u_{|\partial\Omega}|_{L^2(\omega_B)}\right)^\mu,$$

where O is an open subset compactly embedded in Ω. Then, we apply Theorem 8.1, and we find

$$\begin{aligned}\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \lesssim e^{C|\sigma|}\Big(\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \\ + e^{C'|\sigma|}\left(\|u\|_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu}\left(\|P_\sigma u\|_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + |u_{|\partial\Omega}|_{L^2(\omega_B)}\right)^\mu\Big).\end{aligned} \tag{8.11}$$

This, and assumption (8.10) ends the proof. □

8.2 A resolvent estimate

8.2.1 The boundary static case

We use the notation of Section 2.1. We recall that A_δ is the operator defined by:

$$A_\delta := \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta_g & a(x) \end{pmatrix}$$

where a is a function satisfying $a \geq C_I > 0$ on an open subset ω_I of Ω, of domain

$$D(A_\delta) := \{(u_0, u_1) \mid u_0 \in H^2(\Omega),\ u_{0_{|\partial\Omega}} \in H^2(\partial\Omega),\ u_1 \in \mathcal{V}_\delta(\partial\Omega),\ \partial_\nu u_0 + \delta\Sigma u_0 + bu_1 = 0\},$$

where b is a bounded function satisfying $b \geq C_B > 0$ on an open subset ω_B of $\partial\Omega$. In fact, we can allow a or b be identically zero, that is $\omega_I = \emptyset$ of $\omega_B = \emptyset$. We shall prove the first part of Theorem 1.3. Let $U = (u_0, u_1) \in D(A_\delta)$ be such that $(i\sigma \operatorname{Id} + A_\delta)U = F := (f_0, f_1) \in \mathcal{H}_\delta$. This gives

$$i\sigma u_0 - u_1 = f_0, \quad (i\sigma + a(x))u_1 - \Delta_g u_0 = f_1,$$

which is equivalent to

$$u_1 = -f_0 + i\sigma u_0, \quad -\Delta_g u_0 - \sigma^2 u_0 + i\sigma a(x) u_0 = f_1 + a(x) f_0 + i\sigma f_0. \tag{8.12}$$

This yields that u_0 satisfies

$$\begin{cases} (-\Delta_g - \sigma^2) u_0 + i\sigma a(x) u_0 = \tilde{f} & \text{in } \Omega \\ \partial_\nu u_0 + \delta \Sigma u_0 + i\sigma b(x) u_0 = \tilde{g} & \text{in } \partial\Omega \end{cases} \tag{8.13}$$

where $\tilde{f} = f_1 + a(x) f_0 + i\sigma f_0$ and $\tilde{g} = b f_{0|\partial\Omega}$. Observe that in this case, in the definition of S_σ (see the beginning of Section 1.3.2), κ is equal to 0. We multiply the first equation of (8.13) by $\overline{u_0}$ and integrate over Ω, and this yields

$$\begin{aligned} \|\nabla_g u_0\|^2_{L^2(\Omega)} + \delta \langle \Sigma u_0, u_0 \rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} - \sigma^2 \|u_0\|^2_{L^2(\Omega)} + i\sigma\big(a u_0, u_0\big)_{L^2(\Omega)} \\ + i\sigma\big(b u_{0|\partial\Omega}, u_{0|\partial\Omega}\big)_{L^2(\partial\Omega)} = \big(\tilde{f}, u_0\big)_{L^2(\Omega)} + \big(\tilde{g}, u_{0|\partial\Omega}\big)_{L^2(\partial\Omega)}. \end{aligned} \tag{8.14}$$

Taking the imaginary part, we obtain

$$\begin{aligned} C_I |\sigma| \, \|u_0\|^2_{L^2(\omega_I)} + C_B |\sigma| \, |u_{0|\partial\Omega}|^2_{L^2(\partial\Omega)} &\leq |\sigma| \big(a u_0, u_0\big)_{L^2(\Omega)} + |\sigma| \big(b u_{0|\partial\Omega}, u_{0|\partial\Omega}\big)_{L^2(\partial\Omega)} \\ &= \left|\operatorname{Im}\big(\tilde{f}, u_0\big)_{L^2(\Omega)}\right| + \left|\operatorname{Im}\big(\tilde{g}, u_0\big)_{L^2(\partial\Omega)}\right|. \end{aligned} \tag{8.15}$$

Now we apply Theorems 8.1 and 8.5, (since $i\sigma a$ and $i\sigma b$ are low order terms (see Remark 1.6), the Carleman estimates of Theorem 1.5 holds with P_σ and S_σ replaced by $-\Delta_g - \sigma^2 + i\sigma a$ and $\delta\Sigma + i\sigma b$, and Theorems 8.1 and 8.5 applies for u_0), and there exists $C > 0$ such that

$$\|u_0\|_{\mathcal{V}_\delta(\Omega)} \leq C e^{C|\sigma|} \left(\|\tilde{f}\|_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)} + \|u_0\|_{L^2(\omega_I)} + |u_{0|\partial\Omega}|_{L^2(\omega_B)} \right) \tag{8.16}$$

for all $|\sigma| \geq 1$. Using (8.15) and (8.16), we have for $|\sigma| \geq 1$,

$$(C_I + C_B) \|u\|^2_{\mathcal{V}_\delta} \lesssim e^{C|\sigma|} \left(\|\tilde{f}\|^2_{L^2(\Omega)} + |\tilde{g}|^2_{L^2(\partial\Omega)} + \left|\operatorname{Im}\big(\tilde{f}, u_0\big)_{L^2(\Omega)}\right| + \left|\operatorname{Im}\big(\tilde{g}, u_{0|\partial\Omega}\big)_{L^2(\partial\Omega)}\right| \right).$$

The young inequality yields

$$\begin{aligned} \left|\big(\tilde{g}, u_{0|\partial\Omega}\big)_{L^2(\partial\Omega)}\right| + \left|\big(\tilde{f}, u_0\big)_{L^2(\Omega)}\right| \lesssim \varepsilon^{-1} (C_I + C_B)^{-1} e^{C|\sigma|} (\|\tilde{f}\|^2_{L^2(\Omega)} + |\tilde{g}|^2_{L^2(\partial\Omega)}) \\ + \varepsilon (C_I + C_B) e^{-C|\sigma|} (\|u_0\|^2_{L^2(\Omega)} + |u_{0|\partial\Omega}|^2_{L^2(\partial\Omega)}), \end{aligned}$$

for all $\delta' > 0$. Using the trace theorem $|u_{0|\partial\Omega}|_{L^2(\partial\Omega)} \lesssim \|u_0\|_{H^1}$, we obtain for $\varepsilon > 0$ sufficiently small,

$$\|u\|_{\mathcal{V}_\delta(\Omega)} \lesssim e^{C|\sigma|} \left(\|\tilde{f}\|_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)} \right),$$

for $|\sigma| \geq 1$. Using now the definition of $\tilde{f}$ and $\tilde{g}$ below (8.12), we have

$$\begin{aligned} e^{C|\sigma|} \left(\|\tilde{f}\|_{L^2(\Omega)} + |\tilde{g}|^2_{L^2(\partial\Omega)} \right) &\lesssim e^{C|\sigma|} \left(\|f_1\|_{L^2(\Omega)} + (1 + |\sigma|) \|f_0\|_{L^2(\Omega)} + |f_{0|\partial\Omega}|_{L^2(\partial\Omega)} \right) \\ &\lesssim e^{C'|\sigma|} \|(f_0, f_1)\|_{\mathcal{H}_\delta}, \end{aligned}$$

and this ends the proof of (1.18) in Theorem 1.3.

8.2.2 The boundary dynamic case

We now treat the dynamic boundary case, i.e we prove a resolvent estimate for the operator

$$B_\delta := \begin{pmatrix} 0 & -\operatorname{Id} & 0 & 0 \\ -\Delta_g & a & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{Id} \\ \frac{1}{\delta}\gamma_1 & 0 & \Sigma & \frac{1}{\delta}b \end{pmatrix},$$

defined precisely in Section 1.2. Note that a and b are as above, that is, one of them can be identically zero. We shall see that everything we did in the previous case can be done, i.e prove and use the same Carleman estimate. Indeed, let us consider $U := (u_0, u_1, y_0, y_1) \in D(B_\delta)$ and $F := (f_0, f_1, g_0, g_1) \in \mathcal{K}_\delta$, such that

$$(i\sigma \operatorname{Id} + B_\delta)U = F$$

This equality is equivalent to the following system

$$\begin{cases} u_1 &= i\sigma u_0 - f_0 \\ -\Delta_g u_0 - \sigma^2 u_0 + i\sigma a u_0 &= \tilde{f} \\ y_1 &= i\sigma y_0 - g_0 \\ \gamma_1(u_0) + \delta\Sigma y_0 - \delta\sigma^2 y_0 + i\sigma b y_0 &= \tilde{g}, \end{cases} \tag{8.17}$$

where $\tilde{f} = f_1 + (i\sigma + a)f_0$ and $\tilde{g} = \delta g_1 + (\delta i\sigma + b)g_0$. As above, we can multiply the second line by $\overline{u_0}$, integrate by parts, use the transmission condition $u_{0|_{\partial\Omega}} = y_0$, and take the imaginary part to obtain

$$\sigma\Big(a u_0, u_0\Big)_{L^2(\Omega)} + \sigma\Big(b u_{0|_{\partial\Omega}}, u_{0|_{\partial\Omega}}\Big)_{L^2(\Omega)} = \operatorname{Im}\Big(\tilde{f}, u_0\Big)_{L^2(\Omega)} + \operatorname{Im}\Big(\tilde{g}, u_{0|_{\partial\Omega}}\Big)_{L^2(\partial\Omega)}.$$

It is then sufficient to derive a local Carleman estimate for the solution u_0 of the following problem

$$\begin{cases} (-\Delta_g - \sigma^2)u_0 = \tilde{f} \\ \partial_\nu u_0 + \delta(S - \sigma^2)u_0 = \tilde{g}, \end{cases}$$

which corresponds to Theorem 1.5 in the case $\kappa = 1$. This allows us to repeat what is done above, and obtain

$$\|u_0\|_{\mathcal{V}_\delta} \lesssim e^{C|\sigma|}\Big(\|\tilde{f}\|_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)}\Big) \tag{8.18}$$

Using the definitions of $\tilde{f}$ and $\tilde{g}$, and the fact that $\delta \geq 1$, we have

$$\|\tilde{f}\|_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)} \lesssim \|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)} + \delta^{1/2}\Big(|g_0|_{H^1(\partial\Omega)} + |g_1|_{L^2(\Omega)}\Big) \lesssim \|(f_0, f_1, g_0, g_1)\|_{\mathcal{K}_\delta}, \tag{8.19}$$

as $g_0 = f_{0|_{\partial\Omega}}$. From (8.17), we obtain

$$\begin{aligned} &\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \delta^{1/2}\Big(|y_0|_{H^1(\partial\Omega)} + |y_1|_{L^2(\partial\Omega)}\Big) \\ &\lesssim (1 + |\sigma|)\|u_0\|_{H^1(\Omega)} + \|f_0\|_{L^2(\Omega)} + \delta^{1/2}\Big((1 + |\sigma|)|u_{0|_{\partial\Omega}}|_{H^1(\partial\Omega)} + |g_0|_{L^2(\partial\Omega)}\Big) \\ &\lesssim (1 + |\sigma|)\|u_0\|_{\mathcal{V}_\delta} + \|f_0\|_{L^2(\Omega)} + \delta^{1/2}|g_0|_{L^2(\partial\Omega)}. \end{aligned}$$

This, associated with (8.18) and (8.19) ends the proof of Theorem 1.3. □

A Heuristic derivation of the model

In this section, we derive the Ventcel boundary conditions from a transmission problem in an open subset of $\mathbb{R}^n$. Transmission conditions occurs at the interface of a thin layer that surrounds the boundary $\partial\Omega$. The Ventcel boundary condition arises after some approximation. For the sake of simplicity, we consider the case $\Omega = \mathbb{R}^n_+$ with boundary $\partial\Omega = \{x_n = 0\}$. We set $\Omega^\delta = \{x \in \mathbb{R}^n \mid x_n \in (-\delta, 0)\}$, where $\delta > 0$, that describe a layer at the boundary $\partial\Omega$. We also set $\partial\Omega^\delta = \{x_n = -\delta\}$. We then consider the following elliptic problem

$$(-\partial^2_{x_n} - \Delta^T_g)u_1 = f_1 \text{ on } \Omega, \quad (-\partial_{x_n} c \partial_{x_n} - \Delta^T_g)u_2 = f_2 \text{ on } \Omega^\delta, \tag{A.1}$$

where c is a smooth function on Ω^δ, with homogeneous Neumann boundary condition at $\{x_n = -\delta\}$ and transmission condition at $\{x_n = 0\}$

$$\partial_{x_n} u_{2|_{x_n=-\delta}} = 0, \quad u_{1|_{x_n=0}} = u_{2|_{x_n=0}}, \quad \partial_{x_n} u_{1|_{x_n=0}} = c(x', 0)\partial_{x_n} u_{2|_{x_n=0}}. \tag{A.2}$$

Multiplying the second equation of (A.1) by $\frac{1}{\delta}$, and integrating with respect to the variable x_n from $-\delta$ to 0 we obtain

$$-\frac{1}{\delta}(c\partial_{x_n} u_2)_{|_{x_n=0^-}} - \frac{1}{\delta}\int_{-\delta}^{0} \Delta_g^T u_2 dx_n = \frac{1}{\delta}\int_{-\delta}^{0} f_2 dx := g. \tag{A.3}$$

Using the transmission condition and setting $v = \frac{1}{\delta}\int_{-\delta}^{0} u_2 dx_n$, that is averaging u_2 over Ω^δ in the x_n-direction, (A.3) reads

$$-\frac{1}{\delta}\partial_{x_n} u_{1|_{x_n=0}} - \Delta_g^T v = g.$$

If we make the approximation $v \approx u_{1|_{x_n=0}}$ which is meaningfull for $\delta > 0$ small as $u_{1|_{x_n=0^+}} = u_{2|_{x_n=0^-}}$ (we identify the function u_2 and its average on a small domain), we obtain the following transmission condition at $\{x_n = 0^+\}$

$$-\frac{1}{\delta}\partial_{x_n} u_{1|_{x_n=0^+}} - \Delta_g' u_{1|_{x_n=0^+}} = g.$$

The transmission condition at the interface for (A.1) thus yields a Ventcel-type boundary condition for u_1 if we consider the problem from the side of Ω. The approximation $v \approx u_{1|_{x_n=0^+}}$ is only reasonable for a small value of δ. The Ventcel boundary condition can be seen as a good model for a thin layer structure at the boundary.

B Well-posedness results

B.1 Proof of Proposition 2.1.

We first prove that the operator is coercive. For $\lambda \in \mathbb{R}$, the equation $(A_\delta + \lambda I)U = F$, with $U = (u, v) \in D(A_\delta)$ and $F = (f, g) \in \mathcal{H}_\delta$, reads

$$\begin{cases} v &= \lambda u - f \\ Pu + (a+\lambda)\lambda u &= h, \end{cases} \tag{B.1}$$

where $h = g + (a + \lambda) f$, with boundary conditions

$$\partial_\nu u + \delta S u + \lambda b u = \tilde{h} \text{ on } \partial\Omega$$

where $\tilde{h} = b f_{|\partial\Omega}$. We recall that P and S are defined in (1.20). Taking the L^2-inner product of the left hand side of the second line of (B.1) with some $\tilde{u} \in \mathcal{V}_\delta$, we obtain

$$\begin{aligned}
&\Big((P + \lambda(a+\lambda))u, \tilde{u}\Big)_{L^2(\Omega)} \\
= \;& \Big(-\Delta_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} \\
= \;& \Big(\nabla_g u, \nabla_g \tilde{u}\Big)_{L^2(\Omega)} - \Big(\partial_\nu u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} \\
& + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} \\
= \;& \Big(\nabla_g u, \nabla_g \tilde{u}\Big)_{L^2(\Omega)} + \delta \left\langle \Sigma u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} - \Big(\tilde{h}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\
& + \delta\Big(d^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} + \Big(\lambda b u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\
=: \;& a_\delta(u, \tilde{u}) - \Big(\tilde{h}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)}.
\end{aligned}$$

This leads to the following variational problem, for all $\varphi \in \mathcal{V}_\delta$:

$$a_\delta(u, \varphi) = l(\varphi), \tag{B.2}$$

where $l(\varphi) = \int_\Omega h\varphi + \int_{\partial\Omega} \tilde{h}\varphi_{|\partial\Omega}$. Let us prove the coercivity of a_δ for λ large. Recalling that $|.|^2_{L^2(\partial\Omega)} + \langle \Sigma ., . \rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)}$ provides an norm equivalent to $|.|_{H^1(\partial\Omega)}$, by the Young inequality, there exists $C_0 > 0$ such that

$$\begin{aligned}
&\left|\Big(c\nabla_g u, u\Big)_{L^2(\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, u_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \Big(du, u\Big)_{L^2(\Omega)} + \delta\Big(d^T u_{|\partial\Omega}, u_{|\partial\Omega}\Big)_{L^2(\partial\Omega)}\right| \\
&\qquad \leq \|c\|_{L^\infty}\left(\varepsilon^2 \|\nabla_g u\|^2_{L^2(\Omega)} + \varepsilon^{-2}\|u\|^2_{L^2(\Omega)}\right) + \delta C_0 |c^T|_{L^\infty}\Big(\varepsilon^2 \left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} \\
&\qquad\qquad + \varepsilon^{-2}|u_{|\partial\Omega}|^2_{L^2(\partial\Omega)}\Big) + \|d\|_{L^\infty}\|u\|^2_{L^2(\Omega)} + \delta|d^T|_{L^\infty}|u_{|\partial\Omega}|^2_{L^2(\partial\Omega)},
\end{aligned}$$

for all $\varepsilon > 0$, and for all $u \in \mathcal{V}_\delta$. In particular, we choose ε sufficiently small such that

$$1 - \varepsilon^2 \|c\|_{L^\infty} \geq \frac{1}{2} \quad \text{and} \quad 1 - C_0 \varepsilon^2 |c^T|_{L^\infty} \geq \frac{1}{2}, \tag{B.3}$$

and we keep the value of ε fixed in what follows. We shall need the following trace lemma.

Lemma B.1. *For all $u \in H^1(\Omega)$ we have*

$$|u|^2_{L^2(\partial\Omega)} \leq \frac{1}{\varepsilon'^2} \|u\|^2_{L^2(\Omega)} + \varepsilon'^2 \|\nabla_g u\|^2_{L^2(\Omega)},$$

for all $\varepsilon' \in (0, 1]$.

Proof. By locally straightening the boundary, it is sufficient to prove the inequality in the case where $\Omega = \mathbb{R}^n_+$ and $\partial\Omega = \mathbb{R}^n$. We then have for $v \in \overline{C_0^\infty}(\mathbb{R}^n_+)$,

$$u(x', x_n)^2 = -\int_0^{+\infty} \partial_{x_n}(u(x', x_n)^2)\, dx_n = -2 \int_0^{+\infty} u \partial_{x_n} u \, dx_n.$$

Applying the Young inequality yields the result. □

For ε fixed by (B.3), we can now apply Lemma B.1 to have

$$(C_0 \varepsilon^{-2} |c^T|_{L^\infty} + |d^T|_{L^\infty}) |u|^2_{L^2(\partial\Omega)} \leq (C_0 \varepsilon^{-2} |c^T|_{L^\infty} + |d^T|_{L^\infty})(\varepsilon'^2 \|\nabla_g u\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon'^2} \|u\|^2_{L^2(\Omega)}).$$

for all $\varepsilon' > 0$. In particular, we can take ε' such that $\frac{1}{2} - \varepsilon'^2 \left(C_0 \varepsilon^2 |c^T|_{L^\infty} + |d^T|_{L^\infty} \right) \geq \frac{1}{4}$. Hence, there exists $C_1 > 0$ such that

$$\begin{aligned}
a_\delta(u, u) \geq & \|\nabla_g u\|^2_{L^2(\Omega)} (1 - \varepsilon^2 \|c\|_{L^\infty} - \delta \varepsilon'^2 (C_0 \varepsilon^{-2} |c^T|_{L^\infty} + |d^T|_{L^\infty})) \\
& + \|u\|^2_{L^2(\Omega)} (\lambda^2 - \|d\|_{L^\infty} - \varepsilon^{-2} - \delta \varepsilon'^{-2} (C_0 \varepsilon^{-2} |c^T|_{L^\infty} + |d^T|_{L^\infty})) \\
& + \delta \left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} (1 - C_0 \varepsilon^2 \|c^T\|_{L^\infty}) \\
\geq & \frac{1}{4} \|\nabla_g u\|^2_{L^2(\Omega)} + \frac{\delta}{2} \left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + \|u\|^2_{L^2(\Omega)} (\lambda^2 - C_1).
\end{aligned}$$

Therefore, taking λ sufficiently large yield the coercivity of the bilinear form a_δ, uniformly in δ. Let us prove that l is continuous. We have

$$|l(\varphi)| \leq | \int_\Omega (g + (a + \lambda) f) \varphi | + | \int_{\partial\Omega} b f_{|\partial\Omega} \varphi_{|\partial\Omega} | \lesssim \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + |b|_{L^\infty(\partial\Omega)} \|f\|_{H^1(\Omega)} \right) \|\varphi\|_{\mathcal{V}_\delta}.$$

We can then apply the Lax-Milgram theorem to obtain existence and uniqueness of a weak solution $u \in \mathcal{V}_\delta$ of the variational formulation (B.2), and we have the bound

$$\|u\|_{\mathcal{V}_\delta} \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + |b|_{L^\infty(\partial\Omega)} \|f\|_{H^1(\Omega)}. \tag{B.4}$$

In fact, H^2 regularity holds in the interior (see [4]) by standard elliptic theory. It now suffices to prove that for any $x_0 \in \partial\Omega$ there exists a neighborhood V of x_0 in $\overline{\Omega}$ and $\theta \in \overline{C^\infty}(\Omega)$ with $\operatorname{supp} \theta \subset V$ such that $\theta u \in D(\mathcal{A}_\delta)$. In addition, we can impose $\partial_\nu \theta_{|\partial\Omega} = 0$. We set $w = \theta u$. From (B.2), w satisfy for all $\varphi \in \mathcal{V}_\delta(\Omega \cap V)$:

$$a_\delta(w, \varphi) = \int_{\Omega \cap V} t_1 \varphi + \int_{\partial\Omega \cap V} t_2 \varphi. \tag{B.5}$$

where $t_1 := \theta h - \nabla_g \theta \cdot \nabla_g u - \operatorname{div}_g(u \nabla_g \theta) + c \nabla_g \theta u \in L^2(\Omega \cap V)$, and $t_2 := \theta_{|\partial\Omega} \tilde{h} + \delta [S, \theta_{|\partial\Omega}] u_{|\partial\Omega} \in L^2(\partial\Omega \cap V)$. We choose V as in Section 4 and use the local normal geodesic coordinates $x = (x_1, \ldots, x_n)$ described therein. In this coordinates $\partial\Omega = \{x_n = 0\}$ and $\Omega = \mathbb{R}^n_+$. We set $V^0 = V \cap \{x_n = 0\}$. Below, we shall denote $\mathcal{V}(V) := \{u \in H^1(V),\ u_{|x_n=0} \in H^1(V^0)\}$. With θ as above, (B.5) reads

$$\int_V A \nabla w . \nabla \psi + \delta \int_{V^0} A' \nabla^T w_{|_{V^0}} . \nabla^T \psi_{|_{V^0}} = \int_V \tilde{t}_1 \psi + \delta \int_{V_0} \tilde{t}_2 \psi_{|_{V^0}}, \quad \psi \in \mathcal{V}(V), \tag{B.6}$$

where $K, K', k, k' \neq 0$ are bounded functions, and, because of the form of $\tilde{h}$, $\tilde{t}_2$ can be written

$$\tilde{t}_2 = z_1 + \delta z_2 \text{ with } |z_1|_{L^2(\partial\Omega)} \lesssim |b|_{W^{1,\infty}(\partial\Omega)}\|f\|_{H^1(\Omega)}, \text{ and } z_2 \in \delta L^2(\Omega). \tag{B.7}$$

where $A(x)$ (resp. $A'(x)$) is the matrix corresponding to the metric g (resp. $g_{|\partial\Omega}$) satisfying the following ellipticity condition: $\exists C > 0$ independent of x, $\forall \xi \in \mathbb{R}^n$ (resp. $\mathbb{R}^{n-1}$), we have

$$|\xi|^2 \leq CA\xi.\xi \quad (\text{resp.}|\xi'|^2 \leq CA'\xi'.\xi'),$$

where we denoted $\xi = (\xi', \xi_n)$. Let e_k be an element of the cannonical basis of $\mathbb{R}^n$, $k \neq n$, and set $h = |h|e_k$, and $D_h u(x) = \frac{1}{|h|}(u(x+h) - u(x))$, and $|h|$ sufficiently small so that $\text{supp}(w(\cdot + h)) \subset V$. We choose $\psi = D_{-h}D_h w$ in (B.5). We obtain

$$\int_V D_h(A\nabla w) \cdot D_h(\nabla w) + \delta \int_{V_0} D_h(A'\nabla^T w_{|_{V0}}) \cdot D_h \nabla^T w_{|_{V0}} = \int_V \tilde{t}_1 D_{-h} D_h w + \int_{V_0} \tilde{t}_2 D_{-h} D_h w_{|_{V0}}. \tag{B.8}$$

We shall need the following estimation:

$$\|D_h v\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}. \tag{B.9}$$

A proof can be found in [4, Lemma IX.6]. Using (B.9), the right hand side of (B.8) reads

$$\int_V \tilde{t}_1 D_{-h} D_h w \leq \|\tilde{t}_1\|_{L^2(V)} \|\nabla D_h w\|_{L^2(V)} \leq \|\tilde{t}_1\|_{L^2(V)} \|D_h w\|_{H^1(V)},$$

and with (B.7) and trace formula

$$\begin{aligned}\int_{V_0} \tilde{t}_2 D_{-h} D_h w &\lesssim |z_1|_{H^{1/2}(V_0)} |D_{-h} D_h w_{|_{V0}}|_{H^{-1/2}(V_0)} + \delta |z_2|_{L^2(V^0)} |D_{-h} D_h w_{|_{V0}}|_{L^2(V_0)} \\ &\lesssim |b|_{W^{1,\infty}(\partial\Omega)} \|f\|_{H^1(\Omega)} \|D_h w\|_{H^1(V)} + \delta |\tilde{t}_2|_{L^2(V^0)} |D_h w_{|_{V0}}|_{H^1(V^0)}.\end{aligned}$$

Observe that we have the following Leibniz rule

$$D_h(B\nabla\zeta) = B(\cdot + h) D_h \nabla \zeta + (D_h B)\nabla\zeta,$$

for all matrices B and functions ζ. Then, (B.8) yields

$$\tilde{a}_\delta(D_h w, D_h w) \lesssim \left(\|\tilde{t}_1\|_{L^2(V)} + \delta^{1/2}|\tilde{t}_2|_{L^2(V_0)} + |b|_{W^{1,\infty}(\partial\Omega)}\|f\|_{H^1(\Omega)}\right)\|D_h w\|_{\mathcal{V}_\delta},$$

where $\tilde{a}_\delta$ is the bilinear form defined by

$$\tilde{a}_\delta(u, v) = \int_U A(x+h)\nabla u \cdot \nabla v \, dx + \delta \int_{V_0} A'(x+h)\nabla^T u_{|_{V_0}} \cdot \nabla^T v_{|_{V_0}} \, dx'.$$

Observe that $\tilde{a}_\delta$ is coercive with the same argumentation used to prove that a_δ is coercive. Then, we can derive the inequality

$$\|D_h w\|^2_{H^1(V)} + \delta |D_h \nabla^T w|^2_{L^2(U^0)} \lesssim T \|D_h w\|_{\mathcal{V}_\delta}.$$

with $T = \|\tilde{t}_1\|_{L^2(V)} + |\tilde{t}_2|_{L^2(V_0)} + |b|_{W^{1,\infty}(\partial\Omega)}\|f\|_{H^1(\Omega)}$, and then

$$\|D_h w\|_{H^1(V)} + \delta^{1/2} |D_h \nabla^T w|_{L^2(V^0)} \lesssim T. \tag{B.10}$$

Now, for $\psi \in C_0^\infty(V)$, $\tilde{\psi} \in C_0^\infty(V_0)$, and for all $j \in \{1, \ldots, n\}$,$l \in \{1, \ldots, n-1\}$, we have from (B.10)

$$|\int_V w D_{-h} \partial_{x_j} \psi| = |\int_V D_h \partial_{x_j} w \psi| \lesssim T \|\psi\|_{L^2(V)} \tag{B.11}$$

$$|\int_{V^0} w_{|_{V0}} D_{-h} \partial_{x_l} \tilde{\psi}| = |\int_{V^0} D_h \partial_{x_l} w_{|_{V0}} \tilde{\psi}| \lesssim \delta^{-1/2} T |\tilde{\psi}|_{L^2(V^0)}. \tag{B.12}$$

Taking the limit $h \longrightarrow 0$, we obtain $w_{|_{V0}} \in H^2(V_0)$ and $\partial^2_{x_j x_k} w \in L^2(V)$, and $\forall j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, n-1\}$.

It remains to show that $\partial^2_{x_n} w \in L^2(V)$. As we are working with normal geodesic coordinates, the coefficient A of the n-th raw and n-th column is $a_{nn} = 1$. Then (B.6) reads, for $\psi \in C_0^\infty(V)$,

$$\int_V \partial_{x_n} w \partial_{x_n} \psi = \int_V \tilde{t}_1 \psi - \sum_{(k,l)\neq(n,n)} \int_V a_{kl} \partial_{x_k} w \partial_{x_l} \psi.$$

and with (B.11), this yields $|\int_V \partial_n w \partial_n \tilde{\psi}| \leq CT \|\tilde{\psi}\|_{L^2}$. Moreover, $T \lesssim |b|_{W^{1,\infty}(\partial\Omega)}\|f\|_{H^1} + \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}$, since we have (B.4), we have proved $\|u\|^2_{H^2(\Omega)} + \delta |u_{|\partial\Omega}|^2_{H^2(\partial\Omega)} \lesssim |b|_{W^{1,\infty}(\partial\Omega)}\|f\|^2_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}$. Using (B.1), we obtain the sought result. □

B.2 Proof of Proposition 2.5

The proof is essentially the same as in Section B.1. Here, we only prove the existence and uniqueness of the announced variationnal problem in $\mathcal{K}_\delta$. The elliptic regularity can then be proved in the same way as in section in B.1 by the Nirenberg translation method. Let $U = (u_0, u_1, y_0, y_1) \in D(\mathcal{B}_\delta)$ and $F = (f_0, f_1, g_0, g_1) \in \mathcal{K}_\delta$. Then $(\mathcal{B}_\delta + \lambda \,\mathrm{Id})\, U = F$ reads

$$\begin{cases} -u_1 + \lambda u_0 = f_0 \\ P u_0 + (a+\lambda) u_1 = f_1 \\ -y_1 + \lambda y_0 = g_0 \\ \frac{1}{\delta}\partial_\nu u_{0|\partial\Omega} + S y_0 + (\frac{1}{\delta} b + \lambda) y_1 = g_1 \end{cases} \iff \begin{cases} u_1 = \lambda u_0 - f_0 \\ P u_0 + \lambda(a+\lambda) u_0 = h_0 \\ y_1 = \lambda y_0 - g_0 \\ \partial_\nu u_{0|\partial\Omega} + \delta S y_0 + \lambda(b + \delta\lambda) y_0 = h_1, \end{cases}$$

where $h_0 := f_1 + (a+\lambda) f_0$ and $h_1 := \delta g_1 + (b + \delta\lambda) g_0$. Thus, it is sufficient to prove the existence and uniqueness of a solution $(u, y) \in H^1(\Omega) \times H^1(\partial\Omega)$ such that $u_{|\partial\Omega} = y$ of the following system

$$\begin{cases} Pu + \lambda(a+\lambda) u = h_0 \\ \partial_\nu u_{|\partial\Omega} + \delta S y + \lambda(b+\delta\lambda) y = h_1. \end{cases} \tag{B.13}$$

Taking the inner product of the first equation with $\tilde{u}$ yields, with integration by parts,

$$\begin{aligned}
&\Big((P + \lambda(a+\lambda))\, u, \tilde{u} \Big)_{L^2(\Omega)} \\
=&\Big(-\Delta_g u, \tilde{u} \Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u} \Big)_{L^2(\Omega)} + \Big(du, \tilde{u} \Big)_{L^2(\Omega)} + \lambda\Big((a+\lambda) u, \tilde{u} \Big)_{L^2(\Omega)} \\
=&\Big(\nabla_g u, \nabla_g \tilde{u} \Big)_{L^2(\Omega)} + \Big(\lambda(a+\lambda) u, \tilde{u} \Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u} \Big)_{L^2(\Omega)} + \Big(du, \tilde{u} \Big)_{L^2(\Omega)} \\
&+ \delta \left\langle \Sigma u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + \delta\lambda\Big((b+\lambda) u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \Big)_{L^2(\partial\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \Big)_{L^2(\partial\Omega)} \\
&+ \delta\Big(d^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \Big)_{L^2(\partial\Omega)} - \Big(h_1, \tilde{u}_{|\partial\Omega} \Big)_{L^2(\partial\Omega)} \\
:=& b_\delta(u, \tilde{u}) - \Big(h_1, \tilde{u}_{|\partial\Omega} \Big)_{L^2(\partial\Omega)}.
\end{aligned}$$

This leads to the variationnal problem

$$b_\delta(u, \varphi) = l(\varphi),$$

where $l(\varphi) = \int_\Omega h_0 \varphi + \int_{\partial\Omega} h_1 \varphi_{|\partial\Omega}$. Note that $\langle \Sigma u, u \rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + |u|^2_{L^2(\partial\Omega)}$ provides an equivalent norm to $|u|^2_{H^1(\partial\Omega)}$. We now claim that the bilinear form b_δ is coercive. Indeed, observe that as a and b are non-negative functions, we have $(au, u)_{L^2(\Omega)} \geq 0$ and $(bu_{|\partial\Omega}, u_{|\partial\Omega})_{L^2(\partial\Omega)} \geq 0$. We can now apply the Young inequality to obtain

$$\begin{aligned}
\left| \Big(c\nabla_g u, u \Big)_{L^2(\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, u_{|\partial\Omega} \Big)_{L^2(\partial\Omega)} \right| \leq \|c\|_{L^\infty} &\left(\varepsilon^2 \|\nabla_g u\|^2_{L^2(\Omega)} + \varepsilon^{-2} \|u\|^2_{L^2(\Omega)} \right) \\
&+ \delta |c^T|_{L^\infty} \left(\varepsilon^2 C |\nabla_g^T u_{|\partial\Omega}|^2_{L^2(\partial\Omega)} + \varepsilon^{-2} |u_{|\partial\Omega}|^2_{L^2(\Omega)} \right).
\end{aligned} \tag{B.14}$$

for all $\varepsilon > 0$. Taking ε sufficiently small to have $1 - \|c\|_{L^\infty}\delta' \geq \frac{1}{2}$ and $1 - C|c^T|_{L^\infty}\delta' \geq \frac{1}{2}$, and now taking λ sufficiently large, we prove the coercivity of the bilinear form b_δ. It remains to prove that the linear form l is continuous on $\mathcal{V}_\delta$. We have $|\int_\Omega h_0 \varphi| \leq \|h_0\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}$ and because of the form of h_1, we have $|\int_{\partial\Omega} h_1 \varphi_{|\partial\Omega}| \leq |g_0|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)} + \delta |g_1|_{L^2(\Omega)} |\varphi|_{L^2(\partial\Omega)}$. These two estimates yields the continuity of l.

□

C Unique continuation property from the boundary

Theorem C.1. *Let $u \in H^2(\Omega)$ be such that*

$$|Pu(x)| \leq |u(x)| + |\nabla_g u(x)| \text{ a.e on } \Omega, \text{ and } \partial_\nu u_{|\partial\Omega} + S\, u_{|\partial\Omega} = 0 \text{ on } \partial\Omega. \tag{C.1}$$

Assume moreover that $u_{|\partial\Omega} = 0$ on an open subset ω_B of $\partial\Omega$. Then $u = 0$ on Ω.

Proof. We use the normal normal geodesic coordinates introduced in Section 4 in a neighborhood of $y_0 \in \omega_B$. In these coordinates, $y_0 = 0$ and $\partial\Omega = \{x_n = 0\}$. Consider U an open neighborhood of 0 in $\mathbb{R}^n$ such that $U \cap \{x_n = 0\} \subset \omega_B$. Let $x_0 = (x' = 0, -r_0)$, with $r_0 > 0$ to be chosen below. Let $\psi(x) = |x_0 - x|^2$, and define $\varphi(x) = e^{-\lambda\psi(x)}$. Let $r_1 > 0$ and $r_2 > 0$ be such that

- $\{x = (x', x_n) \mid x_n > 0, |x_0 - x| < r_2\} \subset U^+$,
- $r_0 < r_1 < r_2$,

where $U^+ := U \cap \mathbb{R}^n_+$. Observe that for $\lambda > 0$ chosen sufficiently large, φ satisfies the sub-ellipticity condition (4.14). Define the cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$:

$$\begin{cases} \chi(x) = 1 & \text{if } |x_0 - x| < r_1 \\ \chi(x) = 0 & \text{if } |x_0 - x| > r_2. \end{cases}$$

From Proposition 4.3, there exists $\tau_0 > 0$ such that for $\tau \geq \tau_0$

$$\tau^3 \|e^{\tau\varphi}\chi u\|^2_{L^2(U^+)} + \tau\|e^{\tau\varphi} D\chi u\|^2_{L^2(U^+)} \lesssim \|e^{\tau\varphi} P\chi u\|^2_{L^2(U^+)} + \tau|e^{\tau\varphi}\partial_{x_n}\chi u_{|x_n=0}|^2_{L^2(U_0)} + \tau^3|e^{\tau\varphi}\chi u_{|x_n=0}|^2_{L^2(U_0)}, \quad \text{(C.2)}$$

for all $u \in \overline{C_0^\infty}(U^+)$, where $V_0 := V \cap \{x_n = 0\}$. Note that $|e^{\tau\varphi}\chi u_{|x_n=0}|^2_{L^2(U_0)} = 0$ as $U \subset \omega_B$. Observing that $P\chi u = \chi P u + [P, \chi]u$ and $D\chi u = \chi D u + [D, \chi]u$, with assumption (C.1), estimate (C.2) reads

$$\tau^3 \|e^{\tau\varphi}\chi u\|^2_{L^2(V^+)} + \tau\|e^{\tau\varphi}\chi D u\|^2_{L^2(V^+)} \lesssim \|e^{\tau\varphi}\chi u\| + \|e^{\tau\varphi}\chi D u\|^2_{L^2(V^+)} + \|e^{\tau\varphi}[P,\chi]u\|^2_{L^2(V^+)} + \tau|e^{\tau\varphi}[D,\chi]u\|^2_{L^2(V^+)} + \tau|e^{\tau\varphi}\partial_{x_n}\chi u_{|x_n=0}|^2_{L^2(V_0)}.$$

Taking τ sufficiently large, we may ignore the first two terms on the right hand side. Now, observing that the commutators $[P, \chi]$ and $[D, \chi]$ are supported in the region where χ varies, we obtain

$$\|e^{\tau\varphi}[P,\chi]u\|^2_{L^2(V^+)} + \|e^{\tau\varphi}[D,\chi]u\|^2_{L^2(V^+)} \lesssim e^{\tau C_1}\|u\|^2_{H^1(V^+)},$$

with $C_1 = e^{-\lambda r_1}$. Furthermore, note that

$$|e^{\tau\varphi}(\partial_{x_n}\chi u)_{|x_n=0}|^2_{L^2(V_0)} \leq |e^{\tau\varphi}[\partial_{x_n},\chi]_{|x_n=0} u_{|x_n=0}|^2_{L^2(V_0)} + |e^{\tau\varphi}\chi_{|x_n=0} S u_{|x_n=0}|^2_{L^2(V_0)} = 0,$$

by assumption. We then can restrict the left hand side of (C.2) to $W := \left\{x = (x', x_n) \mid x_n > 0, |x_0 - x| < r_1'\right\}$, where r_1' is such that $r_1' < r_1$ and such that $W \neq \emptyset$. This finally yields

$$e^{C_2\tau}\|u\|^2_{H^1(W^+)} \lesssim e^{\tau C_1}\|u\|^2_{H^1(V^+)}, \quad \text{(C.3)}$$

with $C_2 = e^{-\lambda r_1'}$. As $C_1 < C_2$, letting $\tau \longrightarrow +\infty$ in (C.3) yields $u = 0$ in W. Coming back to original coordinates, we have found a non-empty open subset $O \subset \Omega$ such that $u_{|o} = 0$. We can conclude by applying Calderón's unique continuation theorem for elliptic operators of order two. $\square$

C.1 Proof of Proposition 2.8

Below, we shall denote by $\mathcal{H}$ and $\mathcal{V}$ the spaces $\mathcal{V}_{\delta=1}$ and $\mathcal{H}_{\delta=1}$. We recall that $\mathcal{V}_\delta$ and $\mathcal{V}$ (resp. $\mathcal{H}_\delta$ and $\mathcal{H}$) are homeomorphic for each δ. However, only the injection $\mathcal{V}_\delta \hookrightarrow \mathcal{V}$ (resp. $\mathcal{H}_\delta \hookrightarrow \mathcal{H}$) is continuous uniformly in δ. In particular, $\mathcal{V}'_\delta$ and $\mathcal{V}'$ are homeomorphic and only the injection $\mathcal{V}' \hookrightarrow \mathcal{V}'_\delta$ is continuous uniformly in δ. Observe that $f_\delta \rightharpoonup f$ in $L^2(0, T; L^2(\Omega))$ implies $\|f_\delta\|_{L^2(0,T;L^2(\Omega))} \leq C$ uniformly in δ. Consider first $U^0_\delta \in D(A_\delta)$. Multiplying (2.4) by $\partial_t u_\delta$ and integrating by parts over Ω yields

$$\frac{d}{dt}\left(\|\partial_t u_\delta\|^2_{L^2(\Omega)} + \|u_\delta\|^2_{H^1(\Omega)} + \delta|\nabla_g^T u_{\delta|\partial\Omega}|^2_{L^2(\partial\Omega)}\right) + \int_\Omega a|\partial_t u_\delta|^2 + \int_{\partial\Omega} b|\partial_t u_{\delta|\partial\Omega}|^2 = \int_\Omega f_\delta u_\delta.$$

Integrate in time between 0 and t, and using Proposition 2.2,

$$\|\partial_t u_\delta(t)\|^2_{L^2(\Omega)} + \|u_\delta(t)\|^2_{H^1(\Omega)} + \delta|\nabla_g^T u_{\delta|\partial\Omega}(t)|^2_{L^2(\partial\Omega)} + |a^{1/2}|\partial_t u_\delta|^2_{L^2(0,T;L^2(\Omega))} + |b^{1/2}\partial_t u_{\delta|\partial\Omega}|^2_{L^2(0,T;L^2(\partial\Omega))} \leq \|U^0_\delta\|^2_{\mathcal{H}} + \int_0^T \|f_\delta\|^2_{L^2(\Omega)} \leq C, \quad \text{(C.4)}$$

from the assumptions on f_δ (constants may change from one line to an other). Now, consider $w \in L^2(0,T,\mathcal{V})$. Multiplying (2.4) with $U_\delta^0 \in D(A_\delta)$ by w and integrating by parts over $(0,T)\times\Omega$ yields the following variational formulation

$$\int_0^T \left(\partial_t^2 u_\delta, w\right)_{L^2(\Omega)} + \int_0^T \left(\nabla_g u_\delta, \nabla_g w\right)_{L^2(\Omega)} + \delta \int_0^T \left(\nabla_g^T u_{\delta|\partial\Omega}, \nabla_g^T w_{|\partial\Omega}\right)_{L^2(\partial\Omega)}$$
$$+ \int_0^T \left(a\partial_t u_\delta, w\right)_{L^2(\Omega)} + \int_0^T \left(b\partial_t u_{\delta|\partial\Omega}, w_{|\partial\Omega}\right)_{L^2(\partial\Omega)} = \int_0^T \left(f_\delta, w\right)_{L^2(\partial\Omega)}. \quad \text{(C.5)}$$

With (C.4), we have

$$|\int_0^T \left(\partial_t^2 u_\delta, w\right)_{L^2(\Omega)}| \lesssim \int_0^T \left(\|u_\delta\|_{\mathcal{V}_\delta} + \|f_\delta\|_{L^2(\Omega)} + \|a^{1/2}\partial_t u_\delta\|_{L^2(\Omega)} + |b^{1/2}\partial_t u_{\delta|\partial\Omega}|\right)\|w\|_{\mathcal{V}}$$
$$\lesssim (\|U_\delta^0\|_{\mathcal{H}} + \|f_\delta\|_{L^2(0,T,L^2(\Omega))})\|w\|_{\mathcal{V}}.$$

Note that by a density argument that this estimate is still valid for $U_\delta^0 \in \mathcal{H}$. We thus obtain for $U_\delta^0 \in \mathcal{H}$, that $w \mapsto \int_0^T (\partial_t^2 u_\delta, w)_{L^2(\Omega)} \in \mathcal{V}'$ and thus $\partial_t^2 u_\delta \in L^2(0,T;\mathcal{V}')$. Hence,

$$\|\partial_t^2 u_\delta\|^2_{L^2(0,T;\mathcal{V}')} + \|\partial_t u_\delta\|^2_{L^2(\Omega)} + \|u_\delta\|^2_{H^1(\Omega)} + \delta|\nabla_g^T u_{\delta|\partial\Omega}|^2_{L^2(\partial\Omega)} + \|a^{1/2}\partial_t u_\delta\|_{L^2(\Omega)} + |b^{1/2}\partial_t u_{\delta|\partial\Omega}| \le C.$$

This allows to consider a subsequence u_{δ_k} and a function u such that

$$u_{\delta_k} \rightharpoonup u \text{ in } L^2(0,T;H^1(\Omega)), \quad u_{\delta_k} \rightharpoonup u \text{ in } H^1(0,T;L^2(\Omega)),$$
$$u_{\delta_k} \rightharpoonup u \text{ in } H^2(0,T;\mathcal{V}'), \quad b^{1/2}\partial_t u_{\delta_k|\partial\Omega} \rightharpoonup b^{1/2}\partial_t u_{|\partial\Omega} \text{ in } L^2(0,T;L^2(\partial\Omega)).$$

We also can extract a subsequence to u_{δ_k}, denoted again by u_{δ_k} such that $\delta_k^{1/2}\nabla_g^T u_{\delta_k}$ is weakly convergent in $L^2(\partial\Omega)$. In particular, we have $|\nabla_g^T u_{\delta_k}|_{L^2(\Omega)} = O(\delta^{-1/2})$. Taking the limit in (C.5), we thus have

$$\int_0^T \left\langle\partial_t^2 u, w\right\rangle_{\mathcal{V}',\mathcal{V}} + \int_0^T \left(\nabla_g u, \nabla_g w\right)_{L^2(\Omega)} + \int_0^T \left(a\partial_t u, w\right)_{L^2(\Omega)} + \int_0^T \left(b\partial_t u_{|\partial\Omega}, w_{|\partial\Omega}\right)_{L^2(\partial\Omega)}$$
$$= \int_0^T \left(f, w\right)_{L^2(\Omega)}, \quad \text{(C.6)}$$

yielding $| \int_0^T \left\langle\partial_t^2 u, w\right\rangle_{\mathcal{V}',\mathcal{V}} | \lesssim \|w\|_{H^1(\Omega)}$, for all $w \in \mathcal{V}$. As $\mathcal{V}$ is dense in $H^1(\Omega)$ we obtain that $\partial_t^2 u$ can be extended as a linear form defined on $H^1(\Omega)$. We obtain the variational formulation associated with the problem (2.5). We can deduce by existence and uniqueness of the solution of (C.6) that the limit does not depend on the chosen subsequence. This ends the proof. $\square$

D Proof of technical results

D.1 Proof of Proposition 4.2

Observe that the parameter σ does not appear in the Poisson bracket $\{p_2, p_1\}$. We set $\beta(x) = \lambda\tau\varphi(x)$ and $\zeta = \beta^{-1}\xi$. After computations of the Poisson bracket, we obtain

$$\{p_2, p_1\}(x,\xi,\tau) = \beta^3(x)(\lambda\mathcal{R}_1 + \mathcal{R}_2),$$

where $\mathcal{R}_1 = 4p(x, \mathrm{d}_x\psi(x))^2 + (\partial_x\psi\partial_\xi p(x,\zeta))^2$ and

$$\mathcal{R}_2 = 2\partial_\xi p(x,\zeta)\partial_x\tilde{p}(x,\zeta,\mathrm{d}_x\psi(x)) + \mathrm{d}_x^2\psi(x)(\partial_\xi p(x,\zeta), \partial_\xi p(x,\zeta)) - \partial_x p(x,\zeta)\partial_\xi p(x, \mathrm{d}_x\psi(x))$$
$$+ \partial_x p(x, \mathrm{d}_x\psi(x))\partial_\xi p(x, \mathrm{d}_x\psi(x)) + \mathrm{d}_x^2\psi(x)(\partial_\xi p(x, \mathrm{d}_x\psi(x)), \partial_\xi p(x, \mathrm{d}_x\psi(x))). \quad \text{(D.1)}$$

On the one hand, as $\mathrm{d}_x\psi \neq 0$, there exists $C_0 > 0$ such that $\mathcal{R}_1 \ge C_0$, and, on the other hand, we observe that $p_{\varphi,\sigma}(x,\xi,\tau) = 0$, in particular $p_2(x,\xi,\tau,\sigma) = 0$, implies $p(x,\xi) = \sigma^2 + p(x, \tau\mathrm{d}_x\varphi)$, which yields $1 + \frac{\sigma^2}{\beta^2} \lesssim |\zeta|^2 \lesssim 1 + \frac{\sigma^2}{\beta^2}$.

Hence, there exists a constant $C_1 > 0$ such that $|\mathcal{R}_2| \leq C_1 \left(1 + \frac{\sigma^2}{\beta^2}\right)$. Now we choose λ, to be kept fixed in what follows, such that $\lambda C_0 - 2C_1 \geq 1$, and if τ is such that $\dfrac{|\sigma|}{\beta} \leq 1$, we obtain

$$\lambda \mathcal{R}_1 + \mathcal{R}_2 \geq \lambda C_0 - 2C_1 \geq 1 \tag{D.2}$$

and thus $p_{\varphi,\sigma} = 0 \implies \{p_2, p_1\} > 0$. In particular, it holds on the compact set $\{p_{\varphi,\sigma} = 0\} \cap \{\lambda_\tau = 1\}$. We then conclude by homogeneity. Note that the condition $\tau \geq \tilde{c}|\sigma|$ with $\tilde{c} \geq \frac{1}{\lambda \inf \varphi}$ implies $\frac{|\alpha|}{\beta} \leq 1$. □

D.2 Proof of Proposition 4.3

We have

$$\|P_{\varphi,\sigma} v\|^2_{L^2(\mathbb{R}^n_+)} = \|P_2 v\|^2_{L^2(\mathbb{R}^n_+)} + \|P_1 v\|^2_{L^2(\mathbb{R}^n_+)} + i\left(\left(P_1 v, P_2 v\right)_{L^2(\mathbb{R}^n_+)} - \left(P_2 v, P_1 v\right)_{L^2(\mathbb{R}^n_+)}\right).$$

Using the forms of P_1 and P_2 in (4.2) and (4.3) we obtain by integration by parts

$$\begin{aligned} i\left(\left(P_1 v, P_2 v\right)_{L^2(\mathbb{R}^n_+)} - \left(P_2 v, P_1 v\right)_{L^2(\mathbb{R}^n_+)}\right) &= \operatorname{Re}\left(i[P_2, P_1]v, v\right)_{L^2(\mathbb{R}^n_+)} \\ &\quad + \operatorname{Re}\left(P_1 v_{|x_n=0}, D_n v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \operatorname{Re}\left((D_n P_1 - 2\tau(\partial_{x_n}\varphi)P_2)v_{|x_n=0}, v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}. \end{aligned}$$

Using (4.2) and (4.4), the operator $D_n P_1 - 2\tau(\partial_{x_n}\varphi)P_2$ reads

$$\begin{aligned} D_n P_1 - 2\tau(\partial_{x_n}\varphi)P_2 &= 2\tilde{r}(x, \tau \mathrm{d}_{x'}\varphi, D')D_n \\ &\quad - 2\tau\partial_{x_n}\varphi\left(r(x, D') - p(x, \tau \mathrm{d}_x\varphi) - \sigma^2\right) \mod \tau(\mathcal{D}^0 D_n + \mathcal{D}^1_\tau). \end{aligned}$$

Hence

$$i\left(\left(P_1 v, P_2 v\right)_{L^2(\mathbb{R}^n_+)} - \left(P_2 v, P_1 v\right)_{L^2(\mathbb{R}^n_+)}\right) = \operatorname{Re}\left(i[P_2, P_1]v, v\right)_{L^2(\mathbb{R}^n_+)} + \tau \operatorname{Re} \tilde{\mathcal{B}}(v), \tag{D.3}$$

where

$$\begin{aligned} \operatorname{Re} \tilde{\mathcal{B}}(v) &= 2\left(\partial_{x_n}\varphi D_n v_{|x_n=0}, D_n v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + 2\left(\tilde{r}(x, \mathrm{d}_{x'}\varphi, D')v_{|x_n=0}, D_n v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} \\ &\quad + 2\left(\tilde{r}(x, \mathrm{d}_{x'}\varphi, D')D_n v_{|x_n=0}, v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} - 2\left(\partial_{x_n}\varphi\left(r(x, D') - p(x, \tau \mathrm{d}_x\varphi) - \sigma^2\right)v_{|x_n=0}, v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} \\ &\quad + \left(C_0 v_{|x_n=0}, D_n v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \left((\tilde{C}_0 D_n + C_1)v_{|x_n=0}, v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}, \end{aligned} \tag{D.4}$$

with symbols $C_0, \tilde{C}_0 \in \mathcal{D}^0_{T,\tau} = \mathcal{D}^0_\tau$ and $C_1 \in \mathcal{D}^1_{T,\tau}$. Now, we treat the commutator term in (D.3). Its principal symbol is $\{p_2, p_1\}$, a polynomial function of degree 3 in (ξ, τ), and from (4.7), we know that p_1 reads $p_1(x, \xi, \tau) = \tau q_1(x, \xi)$, where q_1 is a polynomial of order one in ξ. Thus, the Poisson bracket reads

$$\{p_2, p_1\} = \tau \xi_n^2 \tilde{b}_0(x, \xi', \tau) + \tau \xi_n \tilde{b}_1(x, \xi', \tau) + \tau \tilde{b}_2(x, \xi', \tau), \tag{D.5}$$

where $\tilde{b}_j$ is a polynomial function of order j in (ξ', τ), $j = 0, 1, 2$. As we imposed $\partial_{x_n}\varphi \geq C' > 0$, using (4.6) and (4.7), we have

$$\xi_n^2 = p_2(x, \xi, \tau, \sigma) + \sigma^2 - r(x, \xi') + \tau^2 \partial_{x_n}\varphi^2 + r(x, \tau \mathrm{d}_{x'}\varphi), \tag{D.6}$$

$$\xi_n = \frac{1}{\partial_{x_n}\varphi}((2\tau)^{-1} p_1(x, \xi, \tau) - \tilde{r}(x, \mathrm{d}_{x'}\varphi, \xi')). \tag{D.7}$$

With (D.6) and (D.7), the bracket (D.5) reads

$$\{p_2, p_1\} = \tau b_0(x) p_2(x, \xi, \tau, \sigma) + b_1(x, \xi', \tau) p_1(x, \xi, \tau) + \tau b_2(x, \xi', \tau, \sigma), \tag{D.8}$$

where b_j are polynomial functions of degree j in (ξ', τ), $j = 0, 1$, and b_2 is a polynomial function of degree 2 in (ξ', τ, σ). We can then write

$$i[P_2, P_1] = \tau b_0 P_2 + \mathrm{Op}_T(b_1) P_1 + \tau \, \mathrm{Op}(c_1') + \tau \, \mathrm{Op}_T(b_2),$$

where $\mathrm{Op}(c_1') \in \mathcal{D}_\tau^1$. This yields

$$\|P_{\varphi,\sigma}v\|^2_{L^2(\mathbb{R}^n_+)} = \|P_2 v\|^2_{L^2(\mathbb{R}^n_+)} + \|P_1 v\|^2_{L^2(\mathbb{R}^n_+)} + \mathrm{Re}\left(\tau\,\mathrm{Op}_T(b_2)v, v\right)_{L^2(\mathbb{R}^n_+)} + \tau\,\mathrm{Re}\,\tilde{\mathcal{B}}(v) + \mathrm{Re}\left(\left(\tau b_0 P_2 + \mathrm{Op}_T(b_1)P_1 + \tau\,\mathrm{Op}(c_1')\right)v, v\right)_{L^2(\mathbb{R}^n_+)}. \tag{D.9}$$

From $(D.8)$ and the sub-ellipticity property, we observe

$$p_2(x,\xi,\tau,\sigma) = p_1(x,\xi,\tau) = 0 \ \Rightarrow\ \tau b_2(x,\xi',\tau,\sigma) \gtrsim \lambda_\tau^3 \gtrsim \lambda_{T,\tau}^3. \tag{D.10}$$

We define the following quantity

$$\gamma(x,\xi',\tau,\sigma) = \left(\frac{\tilde{p}_1(x,\xi',\tau)}{2\tau}\right)^2 + (\partial_{x_n}\varphi)^2\tilde{p}_2(x,\xi',\tau,\sigma), \qquad \mathrm{Op}_T(\gamma) \in \mathcal{D}^2_{T,\tau}, \tag{D.11}$$

where $\tilde{p}_2$ and $\tilde{p}_1$ are defined in (4.10) and (4.11). Observing that $p_1(x,\xi,\tau) = 0$ is equivalent to $\xi_n = -(\partial_{x_n}\varphi)^{-1}\tilde{r}(x,\xi,\mathrm{d}_{x'}\varphi)$, we obtain, for $x \in \overline{V}$, $\xi \in \mathbb{R}^n$, $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$,

$$\begin{cases} \gamma(x,\xi',\tau,\sigma) = 0 \\ \xi_n = -(\partial_{x_n}\varphi)^{-1}\tilde{r}(x,\xi,\mathrm{d}_{x'}\varphi) \end{cases} \Longleftrightarrow p_2(x,\xi,\tau,\sigma) = p_1(x,\xi,\tau) = 0 \Longrightarrow \tau b_2(x,\xi',\tau,\sigma) \gtrsim \lambda^3_{T,\tau}.$$

and finally for $x \in \overline{V}$, $\xi' \in \mathbb{R}^{n-1}$, $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$,

$$\gamma(x,\xi',\tau,\sigma) = 0 \Longrightarrow \tau b_2(x,\xi',\tau,\sigma) \gtrsim \lambda^3_{T,\tau}. \tag{D.12}$$

as both sides of the implication do not involve the variable ξ_n. Moreover, if $\gamma = 0$, taking τ sufficiently large with respect to $|\sigma| \geq 1$, we have $\tau^2 \lesssim |\xi'|^2 \lesssim \tau^2$ and (D.12) yields

$$\gamma(x,\xi',\tau,\sigma) = 0 \Longrightarrow b_2(x,\xi',\tau,\sigma) \gtrsim \lambda^2_{T,\tau}. \tag{D.13}$$

We may now state the following positivity result:

Lemma D.1. *There exists $m_0 > 0$, $\tau_0 > 0$ and $C > 0$ such that*

$$m\lambda_{T,\tau}^{-2}\gamma(x,\xi',\tau,\sigma)^2 + b_2(x,\xi',\tau,\sigma) \geq C\lambda^2_{T,\tau}$$

for all $(x,\xi') \in \overline{V}^+ \times \mathbb{R}^{n-1}$, $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$ and $m \geq m_0$.

The proof is given in Appendix D.3. We can apply the Gårding inequality in the tangential directions to obtain, for $m = m_0$ to remain fixed in what follows,

$$m\,\mathrm{Re}\left(\mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma^2)v, v\right)_{L^2(\mathbb{R}^{n-1})} + \mathrm{Re}\left(\mathrm{Op}_T(b_2)v, v\right)_{L^2(\mathbb{R}^{n-1})} \gtrsim |v|^2_{1,\tau}. \tag{D.14}$$

Then, by symbolic calculus

$$\mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma^2) = \mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma)\,\mathrm{Op}_T(\gamma) \mod \Psi^1_{T,\tau},$$

and thus there exist $c_0 \in \mathcal{S}^0_{T,\tau}$ and $c_1 \in \mathcal{S}^1_{T,\tau}$ such that

$$\left(m\,\mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma^2)v, v\right)_{L^2(\mathbb{R}^{n-1})} = \left(\mathrm{Op}_T(\gamma)v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^{n-1})} + \left(\mathrm{Op}_T(c_1)v, v\right)_{L^2(\mathbb{R}^{n-1})}. \tag{D.15}$$

In terms of the symbols p_2 and p_1, from (D.11), γ reads

$$\begin{aligned}
\gamma(x,\xi',\tau,\sigma) &= (\partial_{x_n}\varphi)^2\tilde{p}_2(x,\xi',\tau,\sigma) + \left(\frac{\tilde{p}_1(x,\xi',\tau)}{2\tau}\right)^2 \\
&= (\partial_{x_n}\varphi)^2\left(p_2(x,\xi,\tau,\sigma) - \xi_n^2\right) + \left(\frac{\tau^{-1}}{2}p_1(x,\xi,\tau) - \xi_n\partial_{x_n}\varphi\right)^2 \\
&= (\partial_{x_n}\varphi)^2 p_2(x,\xi,\tau,\sigma) + \tau^{-1}p_1(x,\xi,\tau)\left(\frac{\tau^{-1}}{4}p_1(x,\xi,\tau) - \xi_n\partial_{x_n}\varphi\right) \\
&= (\partial_{x_n}\varphi)^2 p_2(x,\xi,\tau,\sigma) + \tau^{-1}r_1(x,\xi)p_1(x,\xi,\tau),
\end{aligned}$$

where $r_1(x,\xi) = \dfrac{1}{2}\left(\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi) - \xi_n\partial_{x_n}\varphi\right) \in \mathcal{S}^1_\tau$. We then have

$$\mathrm{Op}_T(\gamma) = \tau^{-1}\,\mathrm{Op}(r_1)P_1 + (\partial_{x_n}\varphi)^2 P_2 + B_1, \tag{D.16}$$

with $B_1 \in \mathcal{D}^1_\tau$. Going back to (D.14), and using (D.15) and (D.16), and integrating in the x_n-variable between 0 and $+\infty$

$$\begin{aligned}\mathrm{Re}\left(\left((\partial_{x_n}\varphi)^2P_2 + \tau^{-1}\,\mathrm{Op}(r_1)P_1 + B_1\right)v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)}\\ + \mathrm{Re}\left(\mathrm{Op}_T(c_1)v, v\right)_{L^2(\mathbb{R}^n_+)} + \mathrm{Re}\left(\mathrm{Op}_T(b_2)v, v\right)_{L^2(\mathbb{R}^n_+)} \gtrsim \|\mathrm{Op}_T(\lambda^1_{T,\tau})v\|^2_{L^2(\mathbb{R}^n_+)}.\end{aligned} \tag{D.17}$$

Thus, (D.9) reads

$$\begin{aligned}\|P_{\varphi,\sigma}v\|^2_{L^2(\mathbb{R}^n_+)} \geq C\tau\|\mathrm{Op}_T(\lambda^1_{T,\tau})v\|_{L^2(\mathbb{R}^n_+)} + \|P_2v\|^2_{L^2(\mathbb{R}^n_+)} + \|P_1v\|^2_{L^2(\mathbb{R}^n_+)} + \tau\,\mathrm{Re}\,\tilde{\mathcal{B}}(v)\\ + \mathrm{Re}\left((\tau b_0P_2 + \mathrm{Op}_T(b_1)P_1 + \tau\,\mathrm{Op}(c_1'))\,v, v\right)_{L^2(\mathbb{R}^n_+)} - \mathrm{Re}\left(\tau\left(\mathrm{Op}_T(c_1)v, v\right)_{L^2(\mathbb{R}^n_+)}\right.\\ \left. + \left(\left(\tau(\partial_{x_n}\varphi)^2P_2 + \mathrm{Op}(r_1)P_1 + \tau B_1\right)v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)}\right).\end{aligned} \tag{D.18}$$

Yet, using the definition of r_1 and integrating by parts with respect to x_n,

$$\begin{aligned}2\,\mathrm{Re}\left(\mathrm{Op}(r_1)P_1v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} = \mathrm{Re}\left(\tilde{r}(x,\mathrm{d}_{x'}\varphi, D')P_1v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)}\\ - \mathrm{Re}\left(P_1v, D_n\partial_{x_n}\varphi\,\mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} + \mathrm{Im}\left(P_1v_{|x_n=0}, \partial_{x_n}\varphi\,\mathrm{Op}_T(c_0)v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}.\end{aligned} \tag{D.19}$$

We set $\mathcal{B}(v) = \tilde{\mathcal{B}}(v) + \frac{1}{2}\tau^{-1}\,\mathrm{Im}\left(P_1v_{|x_n=0}, \partial_{x_n}\varphi\,\mathrm{Op}_T(c_0)v_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}$, and this is precisely the boundary quadratic form stated in the proposition. The end of the proof is devoted to the handling of the remainder terms. Using the Young inequality, we obtain

$$\begin{aligned}\Big|\mathrm{Re}\left(\tau b_0P_2v + \mathrm{Op}_T(b_1)P_1v, v\right)_{L^2(\mathbb{R}^n_+)} - \mathrm{Re}\left(\tau(\partial_{x_n}\varphi)^2P_2v + \frac{1}{2}\tilde{r}(x,D',\mathrm{d}_{x'}\varphi)P_1v + \tau B_1v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)}\\ + \frac{1}{2}\,\mathrm{Re}\left(P_1v, D_n\partial_{x_n}\varphi\,\mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} + \tau\,\mathrm{Re}\left((\mathrm{Op}(c_1') - \mathrm{Op}_T(c_1))\,v, v\right)_{L^2(\mathbb{R}^n_+)}\Big|\\ \lesssim \tau^{-1/2}\left(\|P_2v\|^2_{L^2(R^n_+)} + \|P_1v\|^2_{L^2(\mathbb{R}^n_+)}\right) + \tau^{1/2}\left(\|\mathrm{Op}_T(\lambda^1_{T,\tau})v\|^2_{L^2(\mathbb{R}^n_+)} + \|D_nv\|^2_{L^2(\mathbb{R}^n_+)}\right).\end{aligned} \tag{D.20}$$

Injecting this estimate in (D.18), and taking τ sufficiently large, we obtain

$$\|P_{\varphi,\sigma}v\|^2_{L^2(\mathbb{R}^n_+)} \geq C\left(\tau\|\mathrm{Op}_T(\lambda^1_{T,\tau})v\|^2_{L^2(\mathbb{R}^n_+)} + \|P_1v\|^2_{L^2(\mathbb{R}^n_+)}\right) + \tau\,\mathrm{Re}\,\mathcal{B}(v) - C'\tau^{1/2}\|D_nv\|^2_{L^2(\mathbb{R}^n_+)}.$$

Moreover, we have $\xi_n = \dfrac{\tau^{-1}p_1(x,\xi,\tau)}{2\partial_{x_n}\varphi} - \dfrac{\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi)}{\partial_{x_n}\varphi}$, and this yields

$$\tau\|D_nv\|^2_{L^2(\mathbb{R}^n_+)} \lesssim \tau^{-1}\|P_1v\|^2_{L^2(\mathbb{R}^n_+)} + \tau\|\mathrm{Op}_T(\lambda^1_{T,\tau})v\|^2_{L^2(\mathbb{R}^n_+)},$$

which gives, for τ sufficiently large, the sought result

$$\|P_{\varphi,\sigma}v\|^2_{L^2(\mathbb{R}^n_+)} \geq C\left(\tau\|\mathrm{Op}_T(\lambda_{T,\tau})v\|^2_{L^2(\mathbb{R}^n_+)} + \tau\|D_nv\|^2_{L^2(\mathbb{R}^n_+)}\right) + \tau\,\mathrm{Re}\,\mathcal{B}(v).$$

□

D.3 Proof of Lemma D.1

We set $A_m(x,\xi',\tau,\sigma) = m\lambda^{-2}_{T,\tau}\gamma^2(x,\xi',\tau,\sigma) + b_2(x,\xi',\tau,\sigma)$ with $m > 1$. Observe that A_m is homogeneous of order two in the variable (ξ',τ,σ). We may restrict our analysis to the compact set $L = \overline{V}\times\mathcal{K}$ where $\mathcal{K} := \{(\xi',\tau,\sigma) \in \mathbb{R}^{n-1}\times\mathbb{R}^+\times\mathbb{R}^+,\ |(\xi',\tau,\sigma)| = 1\}$. On L, we have $\gamma^2 \geq 0$, and from (D.13), having $\gamma = 0$ implies $b_2 \geq C > 0$. Observe that A_m is of the form $mf(y) + g(y)$, with y laying in L, and f and g continuous on L. In addition, $f(y) = 0$ implies $g(y) > 0$. Now consider the compact set $E := \{y \in L,\ f(y) = 0\}$. By a continuity argument there exists an open neighborhood F of E such that $\inf_F g > 0$. Then on $L \setminus F$ we have

$$mf(y) + g(y) \geq m\inf f + \inf g > 0$$

by choosing m sufficiently large. This yields $A_m \geq C > 0$ on L. We then conclude by homogeneity.

□

D.4 Proof of Lemma 8.3

By a compactness argument, it suffices to prove the estimate with $\|u\|_{H^1(B(x_0,r_0))}$ in the left hand side, for any x_0 and $r_0 > 0$ such that $B(x_0, r_0) \Subset \Omega$, where $B(x_0, r_0)$ denotes the open ball centered at x_0 with radius r_0.

First, remark that we can assume that u satisfies

$$\|P_\sigma u\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)}, \tag{D.21}$$

otherwise the result is clear. As Ω is connected, we can choose $y_0 \in \omega_I$ and find a continuous path Γ such that

$$\Gamma(0) = x_0 \quad \text{and}\, \Gamma(1) = y_0.$$

Define $r := \dfrac{1}{6}\min(r_1, r_2)$ where $r_1 := \mathrm{d}(\partial\Omega, \Gamma)$ and $r_2 := \mathrm{d}(y_0, \partial\omega_I)$. Now we can define a sequence $(t_j)_j$ by

$$t_0 = 0 \quad \text{and}\, t_{j+1} = \inf E_j, \quad j \ge 0,$$

where $E_j = \{t > t_j \mid \Gamma(t) \notin B(\Gamma(t_j), r)\}$. This sequence is finite by compactness, and then we can define $J := \min\{j \in \mathbb{N},\ E_j = \emptyset\}$ and $t_J = 1$. Then we consider $(x^j)_j$ such that $x^j = \Gamma(t_j)$. Let us assume for a moment that we have the following inequality for some $\mu > 0$ and $C > 0$

$$\|u\|_{H^1(B(x^j,r))} \lesssim e^{C|\sigma|}\|u\|^{1-\mu}_{H^1(\Omega)}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x^{j+1},r))}\right)^\mu \quad 0 \le j \le J-1. \tag{D.22}$$

This, with (D.21) yields

$$\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x^j,r))} \lesssim e^{C|\sigma|}\|u\|^{1-\mu}_{H^1(\Omega)}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x^{j+1},r))}\right)^\mu, \quad 0 \le j \le J-1,$$

and by induction we have, for some $\mu' > 0$ and $C' > 0$

$$\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x_0,r))} \lesssim e^{C'|\sigma|}\|u\|^{1-\mu'}_{H^1(\Omega)}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(y_0,r))}\right)^{\mu'}, \quad 0 \le j \le J-1.$$

As P is elliptic we have the estimate

$$\|u\|_{H^1(B(y_0,r))} \lesssim \|Pu\|_{L^2(\omega_I)} + \|u\|_{L^2(\omega_I)} \le \|P_\sigma u\|_{L^2(\Omega)} + (1+\sigma)\|u\|_{L^2(\omega_I)},$$

and this gives the result. Let us now prove (D.22). We recall that we have the following Carleman estimate away from the boundary (see for instance [13]). Let V be an open bounded subset of $\mathbb{R}^n$, and φ be a weight function satisfying the sub-ellipticity condition in $\overline{V}$. Then, we have

$$\tau^3\|e^{\tau\varphi}u\|^2_{L^2(V)} + \tau\|e^{\tau\varphi}Du\|^2_{L^2(V)} \lesssim \|e^{\tau\varphi}P_\sigma u\|_{L^2(V)}, \tag{D.23}$$

for all $u \in C_0^\infty(V)$ and τ sufficiently large with respect to σ. We shall prove here the following inequality

$$\|u\|_{H^1(B(x^j,3r))} \lesssim e^{C|\sigma|}\|u\|^{1-\mu}_{H^1(\Omega)}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x^j,r))}\right)^\mu.$$

Let us set $\varphi = e^{-\lambda\psi}$ with $\psi(x) = |x - x^j|^2$ and $V = B(x^j, 4r) \setminus B(x^j, r/10)$. Define the cut-off function:

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x - x^j| < r/4 \text{ or } |x - x^j| > 15r/4 \\ 1 & \text{if} \quad 3r/4 < |x - x^j| < 13r/4. \end{cases}$$

Observe that the weight function φ satisfies the sub-ellipticity condition in $\overline{V}$ for λ sufficiently large. We can thus apply (D.23) to χu and this yields

$$\tau^3\|e^{\tau\varphi}\chi u\|^2_{L^2(V)} + \tau\|e^{\tau\varphi}D\chi u\|^2_{L^2(V)} \lesssim \|e^{\tau\varphi}P_\sigma\chi u\|_{L^2(V)}. \tag{D.24}$$

On the right hand side of (D.24), we have

$$\begin{aligned}\|e^{\tau\varphi}P_\sigma\chi u\|_{L^2(V)} &\lesssim \|e^{\tau\varphi}\chi P_\sigma u\|_{L^2(\Omega)} + \|e^{\tau\varphi}[P_\sigma,\chi]u\|_{L^2} \\ &\lesssim e^{\tau C_3}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x^j,r))}\right) + e^{\tau C_1}\|u\|_{H^1(\Omega)},\end{aligned}$$

where $C_3 = e^{-\lambda r/4}$ and $C_1 = e^{-\lambda 13r/4}$. Here, we used that $[P_\sigma, \chi]$ is an differential operator of order 1, supported in the region where χ varies, and that φ decreases as $|x - x^j|$ increases. We can restrict the left hand side of (D.24) to $B(x^j, 3r) \setminus B(x^j, r) =: \tilde{V}$, where $\chi = 1$. This yields

$$\|e^{\tau\varphi}\chi u\|_{L^2(\tilde{V})} + \|e^{\tau\varphi} D\chi u\|_{L^2(\tilde{V})} \gtrsim e^{\tau C_2}\|u\|_{H^1(\tilde{V})},$$

where $C_2 = \varphi(3r)$. Hence

$$\begin{aligned}\|u\|_{H^1(B(x^j,3r))} &\lesssim \|u\|_{H^1(B(x^j,r))} + \|u\|_{H^1(\tilde{V})} \\ &\lesssim e^{\tau(C_3-C_2)}\left(\|P_\sigma u\|_{L^2(\Omega)} + \|u\|_{H^1(B(x^j,r))}\right) + e^{-\tau(C_2-C_1)}\|u\|_{H^1(\Omega)}.\end{aligned} \tag{D.25}$$

Observe that $C_1 < C_2 < C_3$. It remain to apply Lemma 8.4 to conclude the proof. □

D.5 Proof of Lemma 8.4

First observe that $C = 0$ implies $A = 0$, and $B = 0$ also implies $A = 0$ by letting $\tau \to +\infty$. We set $f(\tau) = e^{\beta\tau}B + e^{-\gamma\tau}C$ for all $\tau \in \mathbb{R}$. This function reach its minimum at τ_1 satisfying $e^{\tau_1} = \left(\frac{\gamma C}{\beta B}\right)^{\frac{1}{\beta+\gamma}}$. First assume that $\tau_1 \geq \tilde{\tau}$. Then

$$A \leq e^{\beta\tau_1}B + e^{-\gamma\tau_1}C \leq \left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}} C^{\frac{\beta}{\beta+\gamma}} B^{1-\frac{\beta}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{-\gamma}{\beta+\gamma}} B^{\frac{\gamma}{\beta+\gamma}} C^{1-\frac{\gamma}{\beta+\gamma}} \leq \left(\left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{-\gamma}{\beta+\gamma}}\right) B^{\frac{\gamma}{\beta+\gamma}} C^{\frac{\beta}{\gamma+\beta}}.$$

Second, assume that $\tau_1 < \tilde{\tau}$. We then have $\gamma e^{-\gamma\tilde{\tau}}C \leq \beta e^{\beta\tilde{\tau}}B$. Hence,

$$A \leq C \leq C^{\frac{\beta}{\beta+\gamma}} C^{\frac{\gamma}{\beta+\gamma}} \leq \left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}} e^{\gamma\tilde{\tau}} B^{\frac{\gamma}{\beta+\gamma}} C^{\frac{\beta}{\beta+\gamma}},$$

which gives the result.

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